7

INDETERMINATE FORMS

Art-1. Indeterminate Forms

We have earlier pointed out in the beginning of this chapter that while evaluating limits, we may come across the situations $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 . \infty, 0^0, \infty^0, 1^\infty$, which are called indeterminate forms. For evaluating the form $\frac{0}{0}$, we use L' Hospital's Rule which is given below.

Art-2. L'Hospital's Rule

Statement. If f, g are two functions such that

(*i*) $\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} g(x) = 0$

(*ii*) $f'(x), g'(x)$ both exist and $g'(x) \neq 0 \; \forall x \in (a-\delta, a+\delta),\ \delta > 0$ except possibly at $x = a$.

(*iii*) $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ exists (finitely or infinitely), then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$.

Proof. Let us define two functions F and G such that

$$F(x) = \begin{cases} f(x), \ \forall\, x \in (a-\delta, a+\delta),\ x \neq a \\ 0, \quad x = a \end{cases}$$

$$G(x) = \begin{cases} g(x), \ \forall\, x \in (a-\delta, a+\delta),\ x \neq a \\ 0, \quad x = a \end{cases}$$

Let x be any real number such that $a < x < a + \delta$ then

1. F, G are both continuous in $[a, x]$

$$\left[\because \underset{x \to a}{\text{Lt}} F(x) = \underset{x \to a}{\text{Lt}} f(x) = 0 = F(a), \text{ etc.}\right]$$

2. F, G are both derivable in (a, x)
3. G' is not zero anywhere in (a, x).

$\therefore$ F and G satisfy all the conditions of Cauchy's Mean Value theorem

$\therefore$ there exists atleast one real number $c \in (a, x)$ such that

$$\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(c)}{G'(c)} \quad \text{where } a < c < x$$

or $$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \qquad \ldots(1)$$

Now when $x \to a+,\ c \to a+$

$\therefore$ from (1), we get,

$$\underset{x \to a+}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{c \to a+}{\text{Lt}} \frac{f'(c)}{g'(c)} = \underset{x \to a+}{\text{Lt}} \frac{f'(x)}{g'(x)}$$

$$\therefore \quad \underset{x \to a+}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a+}{\text{Lt}} \frac{f'(x)}{g'(x)} \qquad \ldots(2)$$

Similarly $$\underset{x \to a-}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a-}{\text{Lt}} \frac{f'(x)}{g'(x)} \qquad \ldots(3)$$

From (2) and (3), we get,

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$$

Note 1. If $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ again takes the form $\frac{0}{0}$, we repeat the process or use Taylor's Theorem.

Note 2. L'Hospital's Rule when $x \to \infty$

This rule holds even when $x \to \infty$. Its statement is :

If f, g are two functions such that

(*i*) $\underset{x \to \infty}{\text{Lt}} f(x) = \underset{x \to \infty}{\text{Lt}} g(x) = 0$

(*ii*) $f'(x), g'(x)$ both exist and $g'(x) \neq 0 \;\forall\; x > 0$ except possibly at ∞

(*iii*) $\underset{x \to \infty}{\text{Lt}} \frac{f'(x)}{g'(x)}$ exists, then $\underset{x \to \infty}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to \infty}{\text{Lt}} \frac{f'(x)}{g'(x)}$

Same rule holds when $x \to -\infty$.

Note 3. Standard results $\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1$, $\underset{x \to 0}{\text{Lt}} \frac{\tan x}{x} = 1$, $\underset{x \to 0}{\text{Lt}} (1+x)^{\frac{1}{x}} = e$ etc. should be used before applying L' Hospital's Rule.

Note 4. Sometimes, we shall be using standard expansions in evaluating limits of the form $\frac{0}{0}$. The use of expansions reduces the labour of differentiating time and again. So readers are advised to remember the following expansions. We shall be using these in some of the illustrative examples.

Some Standard Expansions.

(i) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$ (ii) $e^{-x} = 1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \ldots$

(iii) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots$ (iv) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots$

(v) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$ (vi) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{15} - \ldots$

(vii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

(viii) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$

(ix) $(1+x)^{\frac{1}{x}} = e - \frac{ex}{2} + \frac{11ex^2}{24} + \ldots$ (near $x = 0$)

Art-3. Indeterminate Form $\frac{0}{0}$

We give some examples to explain the method.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\underset{x \to 0}{\text{Lt}} \frac{1 - \cos^2 x}{\sin x^2}$

Sol. $\underset{x \to 0}{\text{Lt}} \frac{1 - \cos^2 x}{\sin x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x}{2x \cos x^2}$ [$\because$ of L' Hospital's Rule]

$= \underset{x \to 0}{\text{Lt}} \frac{\sin x \cos x}{x \cos x^2} = \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} \cdot \underset{x \to 0}{\text{Lt}} \frac{\cos x}{\cos x^2} = (1) \cdot \frac{1}{1}$

$\left[\because \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1, \underset{x \to 0}{\text{Lt}} \cos x = 1\right]$

$= 1.$

Example 2. Evaluate $\underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$. (G.N.D.U. 2004)

Sol. $\underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} = \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot \frac{x}{\sin x}$

$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot 1$ $\left(\because \underset{x \to 0}{\text{Lt}} \frac{x}{\sin x} = 1\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{e^x + e^{-x} - \frac{2}{1+x}}{2x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} + \frac{2}{(1+x)^2}}{2} = \frac{1 - 1 + \frac{2}{1}}{2} = \frac{2}{2} = 1.$

Example 3. Evaluate $\underset{x \to 0}{\text{Lt}} \frac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$.

Sol. $\underset{x \to 0}{\text{Lt}} \frac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$

$= \underset{x \to 0}{\text{Lt}} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \left(\frac{x}{\tan x}\right)^2$

$= \underset{x \to 0}{\text{Lt}} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \times (1)^2$ $\left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} = 1\right]$

$= \underset{x \to 0}{\text{Lt}} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{\cos x + \sin x - \frac{1}{1-x}}{3x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{-\sin x + \cos x - \frac{1}{(1-x)^2}}{6x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{-\cos x - \sin x - \frac{2}{(1-x)^3}}{6} = \frac{-1 - 0 - 2}{6} = -\frac{3}{6} = -\frac{1}{2}.$

Example 4. Prove that $\underset{x \to 0}{\text{Lt}} \frac{xe^x - \log(1+x)}{x^2} = \frac{3}{2}$

Sol. $\underset{x\to 0}{\text{Lt}}\ \dfrac{xe^x - \log(1+x)}{x^2}$ $\left[\text{It is of the form } \dfrac{0}{0} \text{ but let us use expansion}\right]$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{x\left(1+x+\frac{x^2}{\lfloor 2}+\ldots\right)-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)}{x^2}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{(x-x)+\left(x^2+\frac{x^2}{2}\right)+\left(\frac{x^3}{2}-\frac{x^3}{3}\right)+\ldots\ldots}{x^2}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\frac{3x^2}{2}+\frac{x^3}{6}+\ldots}{x^2} = \underset{x\to 0}{\text{Lt}}\ \frac{x^2\left(\frac{3}{2}+\frac{x}{6}+\ldots\right)}{x^2}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{x^2\left(\frac{3}{2}+\text{terms containing } x \text{ and its higher powers}\right)}{x^2}$$

$$= \underset{x\to 0}{\text{Lt}}\left(\frac{3}{2}+\text{terms containing } x \text{ and its higher powers}\right)$$

$$= \frac{3}{2}+0 = \frac{3}{2}.$$

Example 5. Evaluate $\underset{x\to 0}{\text{Lt}}\ \dfrac{(1+\sin x)^{\frac{1}{3}}-(1-\sin x)^{\frac{1}{3}}}{x}$

Sol. $\underset{x\to 0}{\text{Lt}}\ \dfrac{(1+\sin x)^{\frac{1}{3}}-(1-\sin x)^{\frac{1}{3}}}{x}$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\left[1+\frac{1}{3}\sin x+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2}\sin^2 x+\ldots\right]-\left[1-\frac{1}{3}\sin x+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2}\sin^2 x+\ldots\right]}{x}$$

$$= \frac{\frac{2}{3}\sin x-\frac{10}{81}\sin^3 x+\ldots\ldots}{x} = \frac{2}{3}\cdot\frac{\sin x}{x}-\frac{10}{81}\cdot\frac{\sin x}{x}\cdot\sin^2 x+\ldots\ldots$$

$$= \frac{2}{3}\cdot 1-\frac{10}{81}\cdot 1\cdot 0+0 = \frac{2}{3}-0 = \frac{2}{3}$$

Example 6. Evaluate the following limits :

(*i*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin ax}{\sin bx}$ (*ii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^x-1}{x}$

(*iii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-1}{b^x-1}$ (*iv*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{(1+x)^n-1}{x}$

(*v*) $\underset{x\to 1}{\text{Lt}}\ \dfrac{\log x}{x-1}$ (*vi*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{x-\tan x}{x-\sin x}$

(*vii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{1-\cos x}{x^2}$ (*viii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{x-\sin x}{x^3}$

(*ix*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^x-e^{-x}-2x}{x-\sin x}$ (*x*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\log(1+x)-x}{1-\cos x}$

(*xi*) $\underset{x\to 0}{\text{Lt}}\ \log(1-x)\cot\dfrac{\pi x}{2}$ (*xii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^x\sin x-x-x^2}{x^2+x\log(1-x)}$

(*xiii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^x\sin x-x-x^2}{x^3}$ (*xiv*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^x-e^{\sin x}}{x-\sin x}$

(*xv*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\log(1-x^2)}{\log\cos x}$ (*xvi*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{x-\sin x}{(x\sin x)^{\frac{3}{2}}}$

(*xvii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{x-\tan^{-1}x}{x-\sin x}$ (*xviii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{x\cos x-\log(1+x)}{x^2}$

(*xix*) $\underset{x\to a}{\text{Lt}}\ \dfrac{a^x-x^a}{a^a-x^x}$ (*xx*) $\underset{x\to 1}{\text{Lt}}\ \dfrac{x^x-x}{1-x+\log x}$.

Sol. (*i*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin ax}{\sin bx}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}}\ \dfrac{a\cos ax}{b\cos bx}$ [∵ of L' Hospital's Rule]

$= \dfrac{a\cos 0}{b\cos 0} = \dfrac{a\times 1}{b\times 1} = \dfrac{a}{b}$

(*ii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^x-1}{x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}}\ \dfrac{e^x-0}{1}$ [∵ of L'Hospital's Rule]

$= \underset{x\to 0}{\text{Lt}}\ e^x = e^0 = 1.$

(*iii*) $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - 1}{b^x - 1}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{a^x \log a}{b^x \log b}$ [∵ of L' Hospital's Rule]

$= \dfrac{a^0 \log a}{b^0 \log b} = \dfrac{\log a}{\log b} = \log_b a.$

(*iv*) $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)^n - 1}{x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{n(1+x)^{n-1}}{1}$ [∵ of L' Hospital's Rule]

$= n(1+0)^{n-1} = n.$

(*v*) $\underset{x \to 1}{\text{Lt}} \dfrac{\log x}{x-1}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 1}{\text{Lt}} \dfrac{\frac{1}{x}}{1}$ [∵ of L' Hospital's Rule]

$= \underset{x \to 1}{\text{Lt}} \dfrac{1}{x} = \dfrac{1}{1} = 1.$

(*vi*) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \tan x}{x - \sin x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{1 - \sec^2 x}{1 - \cos x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{-2 \sec x . \sec x \tan x}{\sin x} = \underset{x \to 0}{\text{Lt}} \dfrac{-\frac{2}{\cos x} \cdot \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}{\sin x}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{-2}{\cos^3 x} = \dfrac{-2}{\cos^3 0} = -\dfrac{2}{1} = -2.$

(*vii*) $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \cos x}{x^2}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{2x} = \dfrac{1}{2} \underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x} = \dfrac{1}{2} \times 1 = \dfrac{1}{2}.$

(*viii*) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \sin x}{x^3}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{1 - \cos x}{3x^2}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{6x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\cos x}{6} = \dfrac{1}{6}.$

(*ix*) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{e^x + e^{-x} - 2}{1 - \cos x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x}}{\sin x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{e^x + e^{-x}}{\cos x} = \dfrac{1+1}{1} = 2.$

(*x*) $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1+x) - x}{1 - \cos x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\frac{1}{1+x} - 1}{\sin x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{-\frac{1}{(1+x)^2}}{\cos x} = \dfrac{-\frac{1}{(1+0)^2}}{1} = -1.$

(*xi*) $\underset{x \to 0}{\text{Lt}} \log(1-x) \cot \dfrac{\pi x}{2}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\log(1-x)}{\tan \frac{\pi x}{2}}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\frac{-1}{1-x}}{\frac{\pi}{2} \sec^2 \frac{\pi x}{2}} = \dfrac{\frac{-1}{1-0}}{\frac{\pi}{2} \sec^2 0} = \dfrac{-1}{\frac{\pi}{2}} = -\dfrac{2}{\pi}.$

(*xii*) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x \cos x + e^x \sin x - 1 - 2x}{2x - \dfrac{x}{1-x} + \log(1-x)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x - 2}{2 - \dfrac{(1-x).1 - x(-1)}{(1-x)^2} - \dfrac{1}{1-x}}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2e^x \cos x - 2}{2 - \dfrac{1}{(1-x)^2} - \dfrac{1}{1-x}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2(e^x \cos x - e^x \sin x)}{-\dfrac{2}{(1-x)^3} - \dfrac{1}{(1-x)^2}} = \frac{2(1-0)}{-2-1} = -\frac{2}{3}.$$

(*xiii*) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x \sin x - x - x^2}{x^3} \qquad \left(\dfrac{0}{0} \text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x \cos x + e^x \sin x - 1 - 2x}{3x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x - 2}{6x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2e^x \cos x - 2}{6x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2(e^x \cos x - e^x \sin x)}{6} = \frac{2(1-0)}{6} = \frac{1}{3}.$$

(*xiv*) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{\sin x}}{x - \sin x} \qquad \left(\dfrac{0}{0} \text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{\sin x} \cos x}{1 - \cos x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{\sin x} \cos^2 x + e^{\sin x} \sin x}{\sin x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{\sin x}.\cos^3 x + e^{\sin x}.2 \cos x \sin x + e^{\sin x} \sin x \cos x + e^{\sin x} \cos x}{\cos x}$$

$$= \frac{1 - 1 + 0 + 0 + 1}{1} = 1.$$

(*xv*) $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1-x^2)}{\log \cos x} \qquad \left(\dfrac{0}{0} \text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{-2x}{1-x^2}}{-\dfrac{\sin x}{\cos x}} = \underset{x \to 0}{\text{Lt}} \frac{\dfrac{-2x}{1-x^2}}{-\tan x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{(1-x^2)(-2) - 2x(-2x)}{(1-x^2)^2}}{-\sec^2 x} = \frac{\dfrac{-2-0}{1}}{-1} = 2.$$

(*xvi*) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \sin x}{(x \sin x)^{\frac{3}{2}}} = \underset{x \to 0}{\text{Lt}} \dfrac{x - \sin x}{x^3 \cdot \left(\dfrac{\sin x}{x}\right)^{\frac{3}{2}}}$

$$= \underset{x \to 0}{\text{Lt}} \frac{x - \sin x}{x^3} \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \cos x}{3x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{6x} = \underset{x \to 0}{\text{Lt}} \frac{\cos x}{6} = \frac{1}{6}.$$

(*xvii*) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \tan^{-1} x}{x - \sin x} \qquad \left(\dfrac{0}{0} \text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \dfrac{1}{1+x^2}}{1 - \cos x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{2x}{(1+x^2)^2}}{\sin x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{(1+x^2)^2.2 - 2x.2(1+x^2).2x}{(1+x^2)^4}}{\cos x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{2(1+x^2) - 8x^2}{(1+x^2)^3}}{\cos x} = \frac{\dfrac{2-0}{1}}{1} = 2.$$

(xviii) $\underset{x\to 0}{\text{Lt}} \dfrac{x\cos x - \log(1+x)}{x^2}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{-x\sin x + \cos x - \dfrac{1}{1+x}}{2x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{-x\cos x - \sin x - \sin x + \dfrac{1}{(1+x)^2}}{2} = \dfrac{-0-0-0+1}{2} = \dfrac{1}{2}.$

(xix) $\underset{x\to a}{\text{Lt}} \dfrac{a^x - x^a}{a^a - x^x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to a}{\text{Lt}} \dfrac{a^x \log a - a x^{a-1}}{0 - x^x(1+\log x)}$

$\left[\text{Put } y = x^x, \therefore \log y = x\log x \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = x\cdot\dfrac{1}{x} + \log x\cdot 1 \Rightarrow \dfrac{dy}{dx} = x^x(1+\log x)\right]$

$= \dfrac{a^a \log a - a\cdot a^{a-1}}{-a^a(1+\log a)} = \dfrac{a^a\log a - a^a}{-a^a(1+\log a)}$

$= \dfrac{\log a - 1}{-(1+\log a)} = -\dfrac{\log a - 1}{1+\log a}.$

(xx) $\underset{x\to 1}{\text{Lt}} \dfrac{x^x - x}{1 - x + \log x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to 1}{\text{Lt}} \dfrac{x^x(1+\log x) - 1}{-1 + \dfrac{1}{x}}$

$\left[\text{Put } y = x^x, \therefore \log y = x\log x \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = x\cdot\dfrac{1}{x} + \log x\cdot 1 \Rightarrow \dfrac{dy}{dx} = x^x(1+\log x)\right]$

$= \underset{x\to 1}{\text{Lt}} \dfrac{x^x \times \dfrac{1}{x} + (1+\log x)\cdot x^x(1+\log x)}{-\dfrac{1}{x^2}}$

$= \dfrac{1\times 1 + (1+0)\times 1(1+0)}{-1} = \dfrac{1+1}{-1} = -2.$

Example 7. Evaluate the following limits :

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{\sinh x - x}{\sin x - x\cos x}$ (ii) $\underset{x\to b}{\text{Lt}} \dfrac{x^b - b^x}{x^x - b^b}$

(iii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - 2\cos x + e^{-x}}{x\sin x}$ (iv) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x + \log\left(\dfrac{1-x}{e}\right)}{\tan x - x}$

Sol. (i) $\underset{x\to 0}{\text{Lt}} \dfrac{\sinh x - x}{\sin x - x\cos x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\cosh x - 1}{\cos x + x\sin x - \cos x}$ [∵ of L'Hospital's Rule]

$= \underset{x\to 0}{\text{Lt}} \dfrac{\cosh x - 1}{x\sin x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\sinh x - 0}{x\cos x + \sin x}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\sinh x}{x\cos x + \sin x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\cosh x}{-x\sin x + \cos x + \cos x} = \dfrac{1}{0+1+1} = \dfrac{1}{2}.$

(ii) $\underset{x\to b}{\text{Lt}} \dfrac{x^b - b^x}{x^x - b^b}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \underset{x\to b}{\text{Lt}} \dfrac{b x^{b-1} - b^x \log b}{x^x(1+\log x) - 0}$ [∵ of L'Hospital's Rule]

$\left[\begin{array}{l}\text{Let } y = x^x, \therefore \log y = \log x^x \\ \Rightarrow \log y = x\log x \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = x\cdot\dfrac{1}{x} + \log x\cdot 1 \\ \Rightarrow \dfrac{dy}{dx} = x^x(1+\log x)\end{array}\right]$

$= \dfrac{b\cdot b^{b-1} - b^b\log b}{b^b(1+\log b)} = \dfrac{b^b - b^b\log b}{b^b(1+\log b)} = \dfrac{1-\log b}{1+\log b}.$

(iii) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - 2\cos x + e^{-x}}{x\sin x} = \underset{x\to 0}{\text{Lt}} \dfrac{e^x - 2\cos x + e^{-x}}{x^2}\cdot\left(\dfrac{x}{\sin x}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - 2\cos x + e^{-x}}{x^2} \qquad \left[\because \underset{\theta \to 0}{\text{Lt}} \frac{\theta}{\sin\theta} = 1\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x + 2\sin x - e^{-x}}{2x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x + 2\cos x + e^{-x}}{2} = \frac{1+2+1}{2} = \frac{4}{2} = 2.$$

(*iv*) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x + \log\left(\frac{1-x}{e}\right)}{\tan x - x} = \underset{x \to 0}{\text{Lt}} \dfrac{e^x + \log(1-x) - \log e}{\tan x - x}$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x + \log(1-x) - 1}{\tan x - x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - \frac{1}{1-x} - 0}{\sec^2 x - 1} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - \frac{1}{(1-x)^2}}{2\sec x . \sec x \tan x - 0} = \underset{x \to 0}{\text{Lt}} \frac{e^x - \frac{1}{(1-x)^2}}{2\sec^2 x \tan x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - \frac{2}{(1-x)^3}}{2[2\sec x . \sec x \tan x . \tan x + \sec^2 x . \sec^2 x]}$$

$$= \frac{1-2}{2[0+1.1]} = \frac{-1}{2} = -\frac{1}{2}.$$

Example 8. Evaluate the following limits :

(*i*) $\underset{x \to \infty}{\text{Lt}} \dfrac{a^{\frac{1}{x}} - b^{\frac{1}{x}}}{\log \frac{x}{x-1}}$ (*ii*) $\underset{x \to \infty}{\text{Lt}} \dfrac{2^{\frac{1}{x}} - 3^{\frac{1}{x}}}{\log \frac{x}{x-1}}$

Sol. (*i*) $\underset{x \to \infty}{\text{Lt}} \dfrac{a^{\frac{1}{x}} - b^{\frac{1}{x}}}{\log \frac{x}{x-1}} = \underset{x \to \infty}{\text{Lt}} \dfrac{a^{\frac{1}{x}} - b^{\frac{1}{x}}}{\log \frac{1}{1-\frac{1}{x}}} \qquad \left(\frac{0}{0} \text{ form}\right)$

$$= \underset{x \to \infty}{\text{Lt}} \frac{a^{\frac{1}{x}} \log a . \left(-\frac{1}{x^2}\right) - b^{\frac{1}{x}} . \log b . \left(-\frac{1}{x^2}\right)}{\frac{1}{\frac{x}{x-1}} . \frac{(x-1).1 - x.1}{(x-1)^2}}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\frac{a^{\frac{1}{x}} \log a}{x^2} - \frac{b^{\frac{1}{x}} \log b}{x^2}}{\frac{1}{x(x-1)}} = \underset{x \to \infty}{\text{Lt}} \frac{a^{\frac{1}{x}} \log a - b^{\frac{1}{x}} \log b}{\frac{x}{x-1}}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{a^{\frac{1}{x}} \log a - b^{\frac{1}{x}} \log b}{\frac{1}{1-\frac{1}{x}}} = \frac{\log a - \log b}{\frac{1}{1-0}} = \log a - \log b = \log\left(\frac{a}{b}\right).$$

(*ii*) Do yourself by taking $a = 2,\ b = 3$.

Example 9. Evaluate the following limits :

(*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{1-\cos x^2}{x^2 \sin x^2}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1+x^3)}{\sin^3 x}$

(*iii*) $\underset{x \to 0}{\text{Lt}} \dfrac{(e^x - 1)\tan^2 x}{x^3}$ (*iv*) $\underset{x \to 0}{\text{Lt}} \dfrac{\cosh x - \cos x}{x \sin x}$

(*v*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sinh x - \sin x}{x \sin^2 x}$

Sol. $\underset{x \to 0}{\text{Lt}} \dfrac{1-\cos x^2}{x^2 \sin x^2} = \underset{x \to 0}{\text{Lt}} \dfrac{1-\cos x^2}{x^4 . \left(\frac{\sin x^2}{x^2}\right)}$

$$= \underset{x \to 0}{\text{Lt}} \frac{1-\cos x^2}{x^4} \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{\sin x^2}{x^2} = 1\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2x . \sin x^2}{4x^3} = \frac{1}{2} \underset{x \to 0}{\text{Lt}} \frac{\sin x^2}{x^2} = \frac{1}{2} \times 1 = \frac{1}{2}.$$

(*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1+x^3)}{\sin^3 x} = \underset{x \to 0}{\text{Lt}} \dfrac{\log(1+x^3)}{x^3 \times \left(\frac{\sin^3 x}{x^3}\right)}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log(1+x^3)}{x^3} \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{3x^2}{1+x^3}}{3x^2} = \underset{x \to 0}{\text{Lt}} \frac{1}{1+x^3} = \frac{1}{1+0} = 1.$$

(iii) $$\underset{x\to 0}{\text{Lt}}\ \frac{(e^x-1)\tan^2 x}{x^3} = \underset{x\to 0}{\text{Lt}}\ \frac{e^x-1}{x}\times\left(\frac{\tan x}{x}\right)^2$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{e^x-1}{x} \qquad \left[\because \underset{x\to 0}{\text{Lt}}\ \frac{\tan x}{x}=1\right]$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{e^x}{1} = \frac{1}{1} = 1.$$

(iv) $$\underset{x\to 0}{\text{Lt}}\ \frac{\cosh x-\cos x}{x\sin x} = \underset{x\to 0}{\text{Lt}}\ \frac{\cosh x-\cos x}{x^2\left(\frac{\sin x}{x}\right)^2}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\cosh x-\cos x}{x^2} \qquad \left[\because \underset{x\to 0}{\text{Lt}}\ \frac{\sin x}{x}=1\right]$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\sinh x+\sin x}{2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\cosh x+\cos x}{2} = \frac{1+1}{2} = \frac{2}{2} = 1.$$

(v) $$\underset{x\to 0}{\text{Lt}}\ \frac{\sinh x-\sin x}{x\sin^2 x} = \underset{x\to 0}{\text{Lt}}\ \frac{\sinh x-\sin x}{x^3\left(\frac{\sin x}{x}\right)^2}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\sinh x-\sin x}{x^3} \qquad \left[\because \underset{x\to 0}{\text{Lt}}\ \frac{\sin x}{x}=1\right]$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\cosh x-\cos x}{3x^2} = \underset{x\to 0}{\text{Lt}}\ \frac{\sinh x-\sin x}{6x}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\cosh x+\cos x}{6} = \frac{1-1}{6} = \frac{2}{6} = \frac{1}{3}.$$

Example 10. Evaluate the following limits :

(i) $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \dfrac{\cos x}{\frac{\pi}{2}-x}$ (ii) $\underset{x\to\pi}{\text{Lt}}\ \dfrac{1-\cos x}{\tan^2 x}$

(iii) $\underset{x\to\pi+}{\text{Lt}}\ \dfrac{\sin x}{\sqrt{x-\pi}}$ (iv) $\underset{x\to 0+}{\text{Lt}}\ \dfrac{3^x-2^x}{\sqrt{x}}$.

Sol. (i) $$\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \frac{\cos x}{\frac{\pi}{2}-x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \frac{-\sin x}{-1} = \underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \sin x = \sin\frac{\pi}{2} = 1.$$

(ii) $$\underset{x\to\pi}{\text{Lt}}\ \frac{1+\cos x}{\tan^2 x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to\pi}{\text{Lt}}\ \frac{-\sin x}{2\tan x\sec^2 x} = \underset{x\to\pi}{\text{Lt}}\ \frac{-\sin x}{\frac{2\sin x}{\cos x}\cdot\frac{1}{\cos^2 x}}$$

$$= -\frac{1}{2}\underset{x\to\pi}{\text{Lt}}\ \cos^3 x = -\frac{1}{2}(-1)^3 = \frac{1}{2}.$$

(iii) $$\underset{x\to\pi+}{\text{Lt}}\ \frac{\sin x}{\sqrt{x-\pi}} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to\pi+}{\text{Lt}}\ \frac{\cos x}{\frac{1}{2\sqrt{x-\pi}}} \qquad [\because \text{ of L' Hospital's Rule}]$$

$$= 2\underset{x\to\pi+}{\text{Lt}}\ \left[\sqrt{x-\pi}\cos x\right] = 0.$$

(iv) $$\underset{x\to 0+}{\text{Lt}}\ \frac{3^x-2^x}{\sqrt{x}} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0+}{\text{Lt}}\ \frac{3^x\log 3-2^x\log 2}{\frac{1}{2\sqrt{x}}} \qquad [\because \text{ of L' Hospital's Rule}]$$

$$= 2\underset{x\to 0+}{\text{Lt}}\ \sqrt{x}\,(3^x\log 3-2^x\log 2) = 0.$$

Example 11. Evaluate the following :

(i) $\underset{x\to 0}{\text{Lt}}\ \dfrac{(1+x)^{\frac{1}{x}}-e}{x}$ (ii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{(1+x)^{\frac{1}{x}}-e+\frac{1}{2}ex}{x^2}$

(iii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{(1+x)^{\frac{1}{x}}-e+\frac{1}{2}ex-\frac{11}{24}ex^2}{x^3}$

Sol. Let $y=(1+x)^{\frac{1}{x}}$

$$\therefore\ \log y = \log(1+x)^{\frac{1}{x}} = \frac{1}{x}\log(1+x)$$

$$= \frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right) = 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots$$

$\therefore \quad \log y = 1 + t$, where $t = -\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots$

$\therefore \quad y = e^{1+t} = e \cdot e^t = e\left[1 + \frac{t}{\lfloor 1} + \frac{t^2}{\lfloor 2} + \frac{t^3}{\lfloor 3} + \dots\right]$

$$\therefore \quad y = e \cdot \left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots\right)^2 + \frac{1}{6}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots\right)^3 + \dots\right]$$

$$= e\left[1 - \frac{1}{2}x + \left(\frac{1}{3} + \frac{1}{8}\right)x^2 + \left(-\frac{1}{4} - \frac{1}{6} - \frac{1}{48}\right)x^3 + \dots\right]$$

$$= e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \dots\right].$$

(*i*) $$\operatorname*{Lt}_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = \operatorname*{Lt}_{x \to 0} \frac{e\left(1 - \frac{1}{2}x + \dots\right) - e}{x}$$

$$= \operatorname*{Lt}_{x \to 0} \left[\frac{-\frac{1}{2}ex + \text{terms contains } x^2 \text{ and higher power of } x}{x}\right]$$

$$= \operatorname*{Lt}_{x \to 0} \left[-\frac{1}{2}e + \text{terms containing } x \text{ and its higher power}\right]$$

$$= -\frac{1}{2}e + 0 = -\frac{1}{2}e.$$

(*ii*) $$\operatorname*{Lt}_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e + \frac{1}{2}ex}{x^2} = \operatorname*{Lt}_{x \to 0} \frac{e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \dots\right) - e + \frac{1}{2}ex}{x^2}$$

$$= \operatorname*{Lt}_{x \to 0} \frac{\frac{11}{24}ex^2 + \dots}{x^2}$$

$$= \operatorname*{Lt}_{x \to 0} \left[\frac{11}{24}e + \text{terms containing } x \text{ and its higher powers}\right]$$

$$= \frac{11}{24} + 0 = \frac{11}{24}.$$

(*iii*) $$\operatorname*{Lt}_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e + \frac{1}{2}ex - \frac{11}{24}ex^2}{x^3}$$

$$= \operatorname*{Lt}_{x \to 0} \frac{e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \dots\right) - e + \frac{1}{2}ex - \frac{11}{24}ex^2}{x^3}$$

$$= \operatorname*{Lt}_{x \to 0} \frac{-\frac{7}{16}ex^3 + \dots}{x^3}$$

$$= \operatorname*{Lt}_{x \to 0} \left[-\frac{7}{16}e + \text{terms containing } x \text{ and its higher powers}\right]$$

$$= -\frac{7}{16}e + 0 = -\frac{7}{16}e.$$

Example 12. Find the values of a and b so that $\operatorname*{Lt}_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3}$ exists and equals 1. (G.N.D.U. 2003, 2008)

Sol. Let $L = \operatorname*{Lt}_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \operatorname*{Lt}_{x \to 0} \frac{x(-a\sin x) + (1 + a\cos x).1 - b\cos x}{3x^2} \quad (\because \text{ of L' Hospital's Rule})$$

$$\therefore \quad L = \operatorname*{Lt}_{x \to 0} \frac{-ax\sin x + 1 + a\cos x - b\cos x}{3x^2} \quad \dots(1)$$

Now denominator of R.H.S. of (1) is zero when $x = 0$. Therefore, in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (1) is also zero at $x = 0$.

$\therefore \quad -a(0)\sin 0 + 1 + a\cos 0 - b\cos 0 = 0$

$\therefore \quad 1 + a - b = 0 \quad \dots(2)$

Assume that $1 + a - b = 0$.

$$\therefore \quad L = \operatorname*{Lt}_{x \to 0} \frac{-ax\sin x + 1 + a\cos x - b\cos x}{3x^2} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \operatorname*{Lt}_{x \to 0} \frac{-a(x\cos x + \sin x.1) - a\sin x + b\sin x}{6x}$$

$$= \operatorname*{Lt}_{x \to 0} \frac{-ax\cos x - 2a\sin x + b\sin x}{6x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-a(1.\cos x - x\sin x) - 2a\cos x + b\cos x}{6}$$

$$= \frac{-a(1.1-0) - 2a(1) + b(1)}{6} = \frac{-a - 2a + b}{6} = \frac{b - 3a}{6}$$

Now L = 1 (given)

$\therefore \quad \frac{b-3a}{6} = 1$

$\Rightarrow \quad b - 3a = 6 \quad \Rightarrow \quad b = 6 + 3a$ ...(3)

Substituting $b = 6 + 3a$ in (2), we get,

$1 + a - 6 - 3a = 0$ or $-2a = 5$

$\therefore \quad a = -\frac{5}{2}$

$\therefore$ from (3), $b = 6 - \frac{15}{2} = -\frac{3}{2}$

$\therefore$ we have $a = -\frac{5}{2}, b = -\frac{3}{2}$.

Example 13. (*i*) If $\underset{x \to 0}{\text{Lt}} \frac{\sin 2x + a\sin x}{x^3}$ be finite, find the value of a and the limit.

(*ii*) Find the values of a and b so that

$$\underset{x \to 0}{\text{Lt}} \frac{x(1 - a\cos x) + b\sin x}{x^3} \text{ exists and equals to } \frac{1}{3}.$$

(*iii*) Given that $\underset{x \to 0}{\text{Lt}} \frac{\sin x + ax + bx^3}{x^3}$ is finite, find a and b.

(*iv*) Determine the values of a, b, c so that $\underset{x \to 0}{\text{Lt}} \frac{x(a + b\cos x) - c\sin x}{x^5}$ exists and equals to 1.

(*v*) Determine the values of a, b, c so that $\underset{x \to 0}{\text{Lt}} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x}$ exists and equals 2. (P.U. 2002)

(*vi*) If $\underset{y \to 0}{\text{Lt}} \frac{ae^y - b\cos y + ce^{-y}}{y\tan y} = 3$, find a, b, c.

Sol. (*i*) Let $L = \underset{x \to 0}{\text{Lt}} \frac{\sin 2x + a\sin x}{x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{2\cos 2x + a\cos x}{3x^2}$ ...(1)

[$\because$ of L' Hospital's Rule]

Now denominator of R.H.S. of (1) is zero when $x = 0$. Therefore in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (1) is also zero at $x = 0$.

$\therefore \quad 2\cos 0 + a\cos 0 = 0 \Rightarrow 2 + a = 0 \Rightarrow a = -2$

Assume that $a = -2$.

$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{2\cos 2x - 2\cos x}{3x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{-4\sin 2x + 2\sin x}{6x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = \frac{-6}{6} = -1$

(*ii*) Let $L = \underset{x \to 0}{\text{Lt}} \frac{x(1 - a\cos x) + b\sin x}{x^3}$. $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{x(a\sin x) + (1 - a\cos x).1 + b\cos x}{3x^2}$

[$\because$ of L' Hospital's Rule]

$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{ax\sin x + 1 - a\cos x + b\cos x}{3x^2}$ ...(1)

Now denominator of R.H.S. of (1) is zero when $x = 0$. Therefore, in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (1) is also zero at $x = 0$.

$\therefore \quad 1 - a + b = 0$ ...(2)

Assume that $1 - a + b = 0$

$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{ax\sin x + 1 - a\cos x + b\cos x}{3x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{a(x\cos x + \sin x) + a\sin x - b\sin x}{6x}$

$= \underset{x \to 0}{\text{Lt}} \frac{ax\cos x + 2a\sin x - b\sin x}{6x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \frac{a\cos x - ax\sin x + 2a\cos x - b\cos x}{6} = \frac{a - 0 + 2a - b}{6} = \frac{3a - b}{6}$

Now $\quad L = \frac{1}{3}$ (given)

$\therefore \quad \frac{3a - b}{6} = \frac{1}{3} \Rightarrow 3a - b = 2$

$\Rightarrow \quad b = 3a - 2$ ...(3)

Substituting $b = 3a - 2$ in (2), we get,

$$1 - a + 3a - 2 = 0 \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

$\therefore$ from (3), $b = \frac{3}{2} - 2 = -\frac{1}{2}$

$\therefore$ we have $a = \frac{1}{2}, b = -\frac{1}{2}$

(iii) Let $L = \underset{x \to 0}{Lt} \frac{\sin x + ax + bx^3}{x^5}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore L = \underset{x \to 0}{Lt} \frac{\cos x + a + 3bx^2}{5x^4}$ ...(1)

[$\because$ of L' Hospital's Rule]

Now denominator of R.H.S. of (1) is zero when $x = 0$. Therefore in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (1) is also zero at $x = 0$

$\therefore 1 + a = 0 \Rightarrow a = -1$

$\therefore$ from (1), $L = \underset{x \to 0}{Lt} \frac{\cos x - 1 + 3bx^2}{5x^4}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore L = \underset{x \to 0}{Lt} \frac{-\sin x + 6bx}{20x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

$\Rightarrow L = \underset{x \to 0}{Lt} \frac{-\cos x + 6b}{60x^2}$ ...(2)

Now denominator of R.H.S. of (2) is zero when $x = 0$. Therefore in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (2) is also zero at $x = 0$.

$\therefore -1 + 6b = 0 \Rightarrow b = \frac{1}{6}$.

(iv) Let $L = \underset{x \to 0}{Lt} \frac{x(a + b\cos x) - c\sin x}{x^5}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{Lt} \frac{x(-b\sin x) + (a + b\cos x).1 - c\cos x}{5x^4}$

$\therefore L = \underset{x \to 0}{Lt} \frac{-bx\sin x + a + b\cos x - c\cos x}{5x^4}$ ...(1)

Now denominator of R.H.S. of (1) is zero when $x = 0$. Therefore in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (1) is also zero at $x = 0$

$\therefore a + b - c = 0$ ...(2)

Now $L = \underset{x \to 0}{Lt} \frac{-b\sin x - bx\cos x - b\sin x + c\sin x}{20x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore L = \underset{x \to 0}{Lt} \frac{-b\cos x - b\cos x + bx\sin x - b\cos x + c\cos x}{60x^2}$

Now denominator of R.H.S. is zero when $x = 0$. Therefore in order that given limit may exist finitely, it is necessary that numerator of R.H.S. is also zero at $x = 0$

$\therefore -b - b - b + c = 0 \Rightarrow c = 3b$ ...(3)

Now $L = \underset{x \to 0}{Lt} \frac{-3b\cos x + bx\sin x + c\cos x}{60x^2}$

$= \underset{x \to 0}{Lt} \frac{3b\sin x + b\sin x + bx\cos x - c\sin x}{120x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{Lt} \frac{3b\cos x + b\cos x - bx\sin x + b\cos x - c\cos x}{120}$

$= \frac{3b + b - 0 + b - c}{120} = \frac{5b - c}{120}$

Now $L = 1$ (given)

$\therefore \frac{5b - c}{120} = 1 \Rightarrow 5b - c = 120 \Rightarrow 5b - 3b = 120$ [$\because$ of (3)]

$\Rightarrow 2b = 120 \Rightarrow b = 60 \Rightarrow c = 180$ [$\because$ of (3)]

$\therefore$ from (2), $a + 60 - 180 = 0 \Rightarrow a = 120$

$\therefore$ we have $a = 120, b = 60, c = 180$

(v) Let $L = \underset{x \to 0}{Lt} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = \underset{x \to 0}{Lt} \frac{ae^x - b\cos x + ce^{-x}}{x^2\left(\frac{\sin x}{x}\right)}$

$\therefore L = \underset{x \to 0}{Lt} \frac{ae^x - b\cos x + ce^{-x}}{x^2}$ ...(1)

Now denominator of R.H.S. of (1) is zero when $x = 0$. Therefore in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (1) is also zero at $x = 0$

$\therefore a - b + c = 0$ ...(2)

$\therefore L = \underset{x \to 0}{Lt} \frac{ae^x + b\sin x - ce^{-x}}{2x}$

Now denominator of R.H.S. of (2) is zero when $x = 0$. Therefore in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (2) is also zero at $x = 0$

$\therefore a - c = 0$ or $a = c$ ...(3)

$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \dfrac{a e^x + b\cos x + c e^{-x}}{2} = \dfrac{a+b+c}{2}$

Now $\quad L = 2$ (given) $\Rightarrow \dfrac{a+b+c}{2} = 2$

$\Rightarrow a + b + c = 4 \qquad \Rightarrow c + b + c = 4$ $\qquad$ [$\because$ of (3)]

$\Rightarrow \qquad b = 4 - 2c$ ...(4)

Putting $a = c$ from (3), $b = 4 - 2c$ from (4) in (2), we get,

$c - 4 + 2c + c = 0 \Rightarrow c = 1$

$\therefore \quad a = 1, \; b = 4 - 2 = 2$

$\therefore$ we have $a = 1, \; b = 2, \; c = 1$.

(*vi*) Let $L = \underset{y \to 0}{\text{Lt}} \dfrac{a e^y - b\cos y + c e^{-y}}{y \tan y} = \underset{y \to 0}{\text{Lt}} \dfrac{a e^y - b\cos y + c e^{-y}}{y^2 \left(\dfrac{\tan y}{y}\right)}$

$\therefore \quad L = \underset{y \to 0}{\text{Lt}} \dfrac{a e^y - b\cos y + c e^{-y}}{y^2}$ ...(1)

Now denominator of R.H.S. of (1) is zero when $y = 0$. Therefore in order than given limit may exist finitely, it is necessary that numerator of R.H.S. of (1) is also zero at $y = 0$.

$\therefore \quad a - b + c = 0$ ...(2)

$\therefore \quad L = \underset{y \to 0}{\text{Lt}} \dfrac{a e^y + b\sin y - c e^{-y}}{2y}$ ...(3)

Now denominator of R.H.S. of (3) is zero when $y = 0$. Therefore, in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (3) is also zero at $y = 0$.

$\therefore \quad a - c = 0$ or $a = c$.(4)

$\therefore \quad L = \underset{y \to 0}{\text{Lt}} \dfrac{a e^y + b\cos y + c e^{-y}}{2} = \dfrac{a+b+c}{2}$

Now $L = 3$ (given) $\Rightarrow \dfrac{a+b+c}{2} = 3$

$\Rightarrow a + b + c = 6 \quad \Rightarrow c + b + c = 6$ $\qquad$ [$\because$ of (4)]

$\Rightarrow b = 6 - 2c$ ...(5)

Putting $a = c$ from (4), $b = 6 - 2c$ from (5) in (2), we get,

$c - 6 + 2c + c = 0 \quad \Rightarrow 4c = 6 \quad \Rightarrow c = \dfrac{3}{2}$

$\therefore \quad a = \dfrac{3}{2}, \; b = 6 - 3 = 3$

$\therefore$ we have $a = \dfrac{3}{2}, \; b = 3, \; c = \dfrac{3}{2}$.

Art-4. Indeterminate Form $\frac{\infty}{\infty}$

If f and g are two differentiable functions in the deleted *nbd.* N of a and $g'(x) \neq 0$ for all x in N, and

(*i*) $\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} g(x) = \infty$

(*ii*) $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ exists, then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$

Note 1. If $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ again assumes the form $\frac{\infty}{\infty}$, then L' Hospital's Rule is repeated till the limit is found.

2. Above result holds when $x \to \infty$ or $x \to -\infty$.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\underset{x \to 0}{\text{Lt}} \frac{\log \sin x}{\cot x}$.

Sol. $\underset{x \to 0}{\text{Lt}} \frac{\log \sin x}{\cot x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{1}{\sin x}.\cos x}{-\operatorname{cosec}^2 x} = -\underset{x\to 0}{\text{Lt}} \sin x \cos x = -\underset{x\to 0}{\text{Lt}} \sin x . \underset{x\to 0}{\text{Lt}} \cos x$$

$$= -0 \times 1 = 0.$$

Example 2. Evaluate $\underset{x\to 0+}{\text{Lt}} \log_{\sin 2x} \sin x.$

Sol. $\underset{x\to 0+}{\text{Lt}} \log_{\sin 2x} \sin x = \underset{x\to 0+}{\text{Lt}} \frac{\log \sin x}{\log \sin 2x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\frac{1}{\sin x}.\cos x}{\frac{1}{\sin 2x}.2\cos 2x} = \underset{x\to 0+}{\text{Lt}} \frac{\cos x}{\sin x} \times \frac{\sin 2x}{2\cos 2x}$$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\cos x}{\sin x} \times \frac{2\sin x \cos x}{2\cos 2x} = \underset{x\to 0+}{\text{Lt}} \frac{\cos^2 x}{\cos 2x} = \frac{1}{1} = 1.$$

Example 3. Evaluate $\underset{x\to \infty}{\text{Lt}} \frac{x^n}{e^x}$, $n \in \text{N}$.

Sol. $\underset{x\to \infty}{\text{Lt}} \frac{x^n}{e^x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \underset{x\to \infty}{\text{Lt}} \frac{n x^{n-1}}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x\to \infty}{\text{Lt}} \frac{n(n-1)x^{n-2}}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

... ...

$$= \underset{x\to \infty}{\text{Lt}} \frac{n(n-1)(n-2)\ldots 2.1}{e^x}$$

$$= \underset{x\to \infty}{\text{Lt}} \frac{\lfloor n}{e^x} = \lfloor n \,.\, \underset{x\to \infty}{\text{Lt}} \frac{1}{e^x} = 0. \quad \left[\because \underset{x\to \infty}{\text{Lt}} \frac{1}{e^x} = \underset{x\to \infty}{\text{Lt}} e^{-x} = 0\right]$$

Example 4. Evaluate the following limits

(*i*) $\underset{x\to \frac{\pi}{2}+}{\text{Lt}} \frac{\tan x}{\tan 3x}$ (*ii*) $\underset{x\to 1}{\text{Lt}} \frac{\log(1-x^2)}{\cot \pi x}$

(*iii*) $\underset{x\to 0}{\text{Lt}} \left(\frac{\log x^2}{\cot^2 x}\right)$ (*iv*) $\underset{x\to 0}{\text{Lt}} \log_{\tan^2 x} \tan^2 2x$.

(H.P.U. 2007)

Sol. (*i*) $\underset{x\to\frac{\pi}{2}+}{\text{Lt}} \dfrac{\tan x}{\tan 3x} = \underset{x\to\frac{\pi}{2}+}{\text{Lt}} \dfrac{\sec^2 x}{3\sec^2 3x}$

$$= \underset{x\to\frac{\pi}{2}+}{\text{Lt}} \frac{\frac{1}{\cos^2 x}}{\frac{3}{\cos^2 3x}} = \frac{1}{3} \underset{x\to\frac{\pi}{2}+}{\text{Lt}} \frac{\cos^2 3x}{\cos^2 x}$$

$$= \frac{1}{3} \underset{x\to\frac{\pi}{2}+}{\text{Lt}} \frac{(4\cos^3 x - 3\cos x)^2}{\cos^2 x}$$

$$= \frac{1}{3} \underset{x\to\frac{\pi}{2}+}{\text{Lt}} (4\cos^2 x - 3)^2 = \frac{1}{3}(4\times 0 - 3)^2 = 3.$$

(*ii*) $\underset{x\to 1}{\text{Lt}} \dfrac{\log(1-x^2)}{\cot \pi x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \underset{x\to 1}{\text{Lt}} \frac{\frac{-2x}{1-x^2}}{-\pi \operatorname{cosec}^2 \pi x} = \frac{2}{\pi} \underset{x\to 1}{\text{Lt}}\, x . \frac{\sin^2 \pi x}{1-x^2}$$

$$= \frac{2}{\pi} \underset{x\to 1}{\text{Lt}}\, x \times \underset{x\to 1}{\text{Lt}} \frac{\sin^2 \pi x}{1-x^2}$$

$$= \frac{2}{\pi} \times 1 \times \underset{x\to 1}{\text{Lt}} \frac{2\pi \sin \pi x \cos \pi x}{-2x} = \frac{2}{\pi} \times 1 \times 0 = 0.$$

(*iii*) $\underset{x\to 0}{\text{Lt}} \dfrac{\log x^2}{\cot^2 x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{2}{x}}{-2\cot x \operatorname{cosec}^2 x} = -\underset{x\to 0}{\text{Lt}} \frac{\sin^3 x}{x\cos x}$$

$$= -\underset{x\to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^3 \times \frac{x^2}{\cos x} = -\underset{x\to 0}{\text{Lt}} \frac{x^2}{\cos x} = -\frac{0}{1} = 0$$

(*iv*) $\underset{x\to 0}{\text{Lt}} \log_{\tan^2 x} \tan^2 2x = \underset{x\to 0}{\text{Lt}} \dfrac{\log \tan^2 2x}{\log \tan^2 x}$

$$= \underset{x\to 0}{\text{Lt}} \frac{\log \tan 2x}{\log \tan x} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{1}{\tan 2x}.2\sec^2 2x}{\frac{1}{\tan x}.\sec^2 x} = 2 \underset{x\to 0}{\text{Lt}} \frac{\sec^2 2x}{\sec^2 x} \times \underset{x\to 0}{\text{Lt}} \frac{\tan x}{\tan 2x}$$

$$= 2 \times \frac{1}{1} \times \underset{x\to 0}{\text{Lt}} \frac{\sec^2 x}{2\sec^2 2x} = 2 \times 1 \times \frac{1}{2} = 1.$$

Example 5. Evaluate the following limits :

(*i*) $\underset{x\to\frac{\pi}{2}+}{\text{Lt}} \dfrac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}$ (*ii*) $\underset{x\to 1-}{\text{Lt}} \log(1-x)\cot\left(\dfrac{\pi x}{2}\right)$

(*iii*) $\underset{x\to a+}{\text{Lt}} \dfrac{\log(x-a)}{\log(e^x - e^a)}$ (*iv*) $\underset{x\to 0+}{\text{Lt}} \log_{\tan x} \tan 2x$

Sol. (*i*) $\underset{x\to\frac{\pi}{2}+}{\text{Lt}} \dfrac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \underset{x\to\frac{\pi}{2}+}{\text{Lt}} \frac{\frac{1}{x-\frac{\pi}{2}}}{\sec^2 x} = \underset{x\to\frac{\pi}{2}+}{\text{Lt}} \frac{\cos^2 x}{x - \frac{\pi}{2}} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \underset{x\to\frac{\pi}{2}+}{\text{Lt}} \left(\frac{-2\cos x \sin x}{1}\right) = -2\cos\frac{\pi}{2}\sin\frac{\pi}{2} = 0.$$

(*ii*) $\underset{x\to 1-}{\text{Lt}} \log(1-x)\cot\dfrac{\pi x}{2} = \underset{x\to 1-}{\text{Lt}} \dfrac{\log(1-x)}{\tan\frac{\pi x}{2}}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \underset{x\to 1-}{\text{Lt}} \frac{\frac{-1}{1-x}}{\frac{\pi}{2}\sec^2\frac{\pi x}{2}} = -\frac{2}{\pi} \underset{x\to 1-}{\text{Lt}} \frac{\cos^2\frac{\pi x}{2}}{1-x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= -\frac{2}{\pi} \underset{x\to 1-}{\text{Lt}} \frac{-2.\cos\frac{\pi x}{2}\sin\frac{\pi x}{2}.\left(\frac{\pi}{2}\right)}{-1}$$

$$= -2 \underset{x\to 1-}{\text{Lt}} \cos\frac{\pi x}{2}\sin\frac{\pi x}{2} = -2\times\cos\frac{\pi}{2}\sin\frac{\pi}{2}$$

$$= 0.$$

(*iii*) $\displaystyle \operatorname*{Lt}_{x \to a+} \frac{\log (x-a)}{\log (e^x - e^a)}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \operatorname*{Lt}_{x \to a+} \frac{\dfrac{1}{x-a}}{\dfrac{e^x}{e^x - e^a}} = \operatorname*{Lt}_{x \to a+} \frac{e^x - e^a}{(x-a)e^x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \operatorname*{Lt}_{x \to a+} \frac{e^x}{(x-a)e^x + e^x} = \frac{e^a}{0 + e^a} = 1.$$

(*iv*) $\displaystyle \operatorname*{Lt}_{x \to 0+} \log_{\tan x} \tan 2x = \operatorname*{Lt}_{x \to 0+} \frac{\log \tan 2x}{\log \tan x}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \operatorname*{Lt}_{x \to 0+} \frac{\dfrac{1}{\tan 2x} . 2 \sec^2 2x}{\dfrac{1}{\tan x} . \sec^2 x} = 2 . \operatorname*{Lt}_{x \to 0+} \frac{\sec^2 2x}{\sec^2 x} \operatorname*{Lt}_{x \to 0+} \frac{\tan x}{\tan 2x}$$

$$= 2 \times 1 \times \operatorname*{Lt}_{x \to 0+} \frac{\sec^2 x}{2 \sec^2 2x} = 2 \times 1 \times \frac{1}{2} = 1.$$

Example 6. Evaluate the following limits :

(*i*) $\displaystyle \operatorname*{Lt}_{x \to \infty} \frac{\log x}{x}$ (*ii*) $\displaystyle \operatorname*{Lt}_{x \to \infty} \frac{\log x}{x^n}$, $n \in \mathbf{N}$.

Sol. (*i*) $\displaystyle \operatorname*{Lt}_{x \to \infty} \frac{\log x}{x}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \operatorname*{Lt}_{x \to \infty} \frac{\dfrac{1}{x}}{1} = \operatorname*{Lt}_{x \to \infty} \frac{1}{x} = 0.$$

(*ii*) $\displaystyle \operatorname*{Lt}_{x \to \infty} \frac{\log x}{x^n}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \operatorname*{Lt}_{x \to \infty} \frac{\dfrac{1}{x}}{n x^{n-1}} = \frac{1}{n} \operatorname*{Lt}_{x \to \infty} \frac{1}{x^n} = 0.$$

Art-5. Indeterminate Form $\infty - \infty$

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$, then to determine $\lim_{x \to a} [f(x) - g(x)]$, we write

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}$$ which is of the form $\frac{0}{0}$ as $x \to a$ and can be evaluated

by using L' Hospital's Rule.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$.

Sol. $\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$ $(\infty - \infty \text{ form})$

$$= \lim_{x \to 0} \left[\frac{x - e^x + 1}{x(e^x - 1)} \right] \qquad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \to 0} \left[\frac{1 - e^x + 0}{x . e^x + (e^x - 1) . 1} \right] = \lim_{x \to 0} \frac{1 - e^x}{x e^x + e^x - 1} \qquad \left(\frac{0}{0} \text{ form} \right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-e^x}{xe^x + e^x + e^x} = -\frac{1}{0+1+1} = -\frac{1}{2}.$$

Example 2. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\cot^2 x - \frac{1}{x^2}\right)$.

(P.U. 2001; G.N.D.U. 2007, 2009; H.P.U. 2008)

Sol. $\underset{x \to 0}{\text{Lt}} \left(\cot^2 x - \frac{1}{x^2}\right)$ $(\infty - \infty \text{ form})$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{1}{\tan^2 x} - \frac{1}{x^2}\right) = \underset{x \to 0}{\text{Lt}} \frac{x^2 - \tan^2 x}{x^2 \tan^2 x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 - \tan^2 x}{x^4} \cdot \left(\frac{x}{\tan x}\right)^2$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 - \tan^2 x}{x^4} \cdot (1)^2. \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} = 1\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 - \tan^2 x}{x^4} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2x - 2\tan x \sec^2 x}{4x^3} = \underset{x \to 0}{\text{Lt}} \frac{2x - 2\tan x(1 + \tan^2 x)}{4x^3}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x - \tan x - \tan^3 x}{2x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \sec^2 x - 3\tan^2 x \sec^2 x}{6x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - (1 + \tan^2 x) - 3\tan^2 x(1 + \tan^2 x)}{6x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - 1 - \tan^2 x - 3\tan^2 x - 3\tan^4 x}{6x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-4\tan^2 x - 3\tan^4 x}{6x^2} = -\underset{x \to 0}{\text{Lt}} \frac{4 + 3\tan^2 x}{6} \cdot \left(\frac{\tan x}{x}\right)^2$$

$$= -\frac{4+0}{6} \cdot (1)^2 = -\frac{2}{3}.$$

Example 3. Evaluate the following limits :

(*i*) $\underset{x \to 1}{\text{Lt}} \left(\frac{1}{x^2 - 1} - \frac{2}{x^4 - 1}\right)$. (*ii*) $\underset{x \to 0}{\text{Lt}} \left[\frac{1}{2x} - \frac{1}{x(e^{\pi x} + 1)}\right]$

(*iii*) $\underset{x \to 2}{\text{Lt}} \left[\frac{1}{\log(x-1)} - \frac{1}{x-2}\right]$ (*iv*) $\underset{x \to 1}{\text{Lt}} \left(\frac{x}{x-1} - \frac{1}{\log x}\right)$

(*v*) $\underset{x \to 0}{\text{Lt}} \left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right]$.

Sol. (*i*) $\underset{x \to 1}{\text{Lt}} \left(\frac{1}{x^2 - 1} - \frac{2}{x^4 - 1}\right)$ $(\infty - \infty \text{ form})$

$$= \underset{x \to 1}{\text{Lt}} \left[\frac{1}{x^2 - 1} - \frac{2}{(x^2 - 1)(x^2 + 1)}\right] = \underset{x \to 1}{\text{Lt}} \left[\frac{x^2 + 1 - 2}{(x^2 - 1)(x^2 + 1)}\right]$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^2 - 1}{(x^2 - 1)(x^2 + 1)} = \underset{x \to 1}{\text{Lt}} \frac{1}{x^2 + 1} = \frac{1}{1+1} = \frac{1}{2}.$$

(*ii*) $\underset{x \to 0}{\text{Lt}} \left[\frac{1}{2x} - \frac{1}{x(e^{\pi x} + 1)}\right]$ $(\infty - \infty \text{ form})$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^{\pi x} + 1 - 2}{2x(e^{\pi x} + 1)} = \underset{x \to 0}{\text{Lt}} \frac{e^{\pi x} - 1}{2x(e^{\pi x} + 1)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^{\pi x} \cdot \pi}{2x \cdot (e^{\pi x} \cdot \pi) + (e^{\pi x} + 1) \cdot 2} = \frac{\pi}{0 + (1+1)(2)} = \frac{\pi}{4}.$$

(*iii*) $\underset{x \to 2}{\text{Lt}} \left[\frac{1}{\log(x-1)} - \frac{1}{x-2}\right]$ $[\infty - \infty \text{ form}]$

$$= \underset{x \to 2}{\text{Lt}} \frac{x - 2 - \log(x-1)}{(x-2)\log(x-1)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 2}{\text{Lt}} \frac{1 - \frac{1}{x-1}}{\frac{x-2}{x-1} + \log(x-1)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 2}{\text{Lt}} \frac{\frac{1}{(x-1)^2}}{\frac{(x-1).1 - (x-2).1}{(x-1)^2} + \frac{1}{x-1}} = \underset{x \to 2}{\text{Lt}} \frac{\frac{1}{(x-1)^2}}{\frac{1}{(x-1)^2} + \frac{1}{x-1}} = \frac{1}{1+1} = \frac{1}{2}$$

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(iv) $\underset{x\to 1}{\text{Lt}}\left(\dfrac{x}{x-1}-\dfrac{1}{\log x}\right)$ $(\infty-\infty$ form)

$$= \underset{x\to 1}{\text{Lt}}\ \frac{x\log x - x + 1}{(x-1)\log x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 1}{\text{Lt}}\ \frac{x.\frac{1}{x}+\log x.1-1}{(x-1)\frac{1}{x}+\log x.1} = \underset{x\to 1}{\text{Lt}}\ \frac{\log x}{\frac{x-1}{x}+\log x}$$

$$= \underset{x\to 1}{\text{Lt}}\ \frac{\frac{1}{x}}{\frac{x.1-(x-1).1}{x^2}+\frac{1}{x}} = \frac{1}{1+1} = \frac{1}{2}$$

(v) $\underset{x\to 0}{\text{Lt}}\left[\dfrac{1}{x}-\dfrac{1}{x^2}\log(1+x)\right]$ $(\infty-\infty$ form)

$$= \underset{x\to 0}{\text{Lt}}\left[\frac{x-\log(1+x)}{x^2}\right] \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{1-\frac{1}{1+x}}{2x} = \underset{x\to 0}{\text{Lt}}\ \frac{\frac{1}{(1+x)^2}}{2} = \frac{1}{2}.$$

Example 4. Evaluate the following limits :

(i) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right)$ (G.N.D.U. 2008) (ii) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$

(iii) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x}-\cot x\right)$ (iv) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x^2}-\dfrac{\cot x}{x}\right)$

(v) $\underset{x\to \frac{\pi}{2}}{\text{Lt}}\ (\sec x-\tan x)$ (vi) $\underset{x\to 0}{\text{Lt}}\ (\operatorname{cosec} x-\cot x)$.

Sol. (i) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right)$ $(\infty-\infty$ form)

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\sin^2 x - x^2}{x^2.\sin^2 x} = \underset{x\to 0}{\text{Lt}}\ \frac{\sin^2 x - x^2}{x^4.\left(\frac{\sin x}{x}\right)^2}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\sin^2 x - x^2}{x^4} \qquad \left[\because \underset{x\to 0}{\text{Lt}}\ \frac{\sin x}{x}=1\right]$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{2\sin x\cos x-2x}{4x^3} = \underset{x\to 0}{\text{Lt}}\ \frac{\sin 2x-2x}{4x^3} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{2\cos 2x-2}{12x^2} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{-4\sin 2x}{24x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{-8\cos 2x}{24} = \frac{-8\times 1}{24} = -\frac{1}{3}.$$

(ii) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$ $(\infty-\infty$ form)

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\sin x-x}{x\sin x} = \underset{x\to 0}{\text{Lt}}\ \frac{\sin x-x}{x^2.\left(\frac{\sin x}{x}\right)} = \underset{x\to 0}{\text{Lt}}\ \frac{\sin x-x}{x^2} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\cos x-1}{2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{-\sin x}{2} = \frac{-0}{2} = 0.$$

(iii) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x}-\cot x\right)$ $(\infty-\infty$ form)

$$= \underset{x\to 0}{\text{Lt}}\left(\frac{1}{x}-\frac{1}{\tan x}\right) = \underset{x\to 0}{\text{Lt}}\ \frac{\tan x-x}{x\tan x}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\tan x-x}{x^2\left(\frac{\tan x}{x}\right)} = \underset{x\to 0}{\text{Lt}}\ \frac{\tan x-x}{x^2} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\sec^2 x-1}{2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{2\sec x.\sec x\tan x}{2} = \frac{2\sec 0.\sec 0.\tan 0}{2} = \frac{2\times 1\times 1\times 0}{2} = 0.$$

(iv) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{1}{x^2}-\dfrac{\cot x}{x}\right)$ $(\infty-\infty$ form)

$$= \underset{x\to 0}{\text{Lt}}\left(\frac{1}{x^2}-\frac{1}{x\tan x}\right) = \underset{x\to 0}{\text{Lt}}\ \frac{\tan x-x}{x^2\tan x}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\tan x-x}{x^3\left(\frac{\tan x}{x}\right)} = \underset{x\to 0}{\text{Lt}}\ \frac{\tan x-x}{x^3} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\sec^2 x-1}{3x^2} = \underset{x\to 0}{\text{Lt}}\ \frac{\tan^2 x}{3x^2} = \frac{1}{3}\underset{x\to 0}{\text{Lt}}\left(\frac{\tan x}{x}\right)^2 = \frac{1}{3}\times(1)^2 = \frac{1}{3}.$$

(v) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sec x - \tan x)$ $(\infty - \infty \text{ form})$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(\frac{1 - \sin x}{\cos x}\right) \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{-\cos x}{-\sin x} = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\cos x}{\sin x} = \frac{0}{1} = 0.$$

(vi) $\underset{x \to 0}{\text{Lt}} (\text{cosec}\, x - \cot x)$ $(\infty - \infty \text{ form})$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \underset{x \to 0}{\text{Lt}} \frac{1 - \cos x}{\sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{\cos x} = \frac{0}{1} = 0.$$

Example 5. Evaluate $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(x \tan x - \frac{\pi}{2} \sec x\right)$

Sol. $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(x \tan x - \frac{\pi}{2} \sec x\right)$ $(\infty - \infty \text{ form})$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(\frac{x \sin x}{\cos x} - \frac{\pi}{2\cos x}\right) = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(\frac{2x \sin x - \pi}{2 \cos x}\right) \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(\frac{2x \cos x + 2 \sin x}{-2 \sin x}\right) = \frac{2 \cdot \frac{\pi}{2} \cdot \cos\frac{\pi}{2} + 2 \sin\frac{\pi}{2}}{-2 \sin\frac{\pi}{2}} = \frac{\pi \times 0 + 2 \times 1}{-2 \times 1}$$

$$= \frac{2}{-2} = -1$$

Art-6. Indeterminate Form 0 . ∞

If $\underset{x \to a}{\text{Lt}} f(x) = 0$ and $\underset{x \to a}{\text{Lt}} g(x) = \infty$, then to find $\underset{x \to a}{\text{Lt}} f(x) g(x)$ we write $f(x)$ $g(x) = \dfrac{f(x)}{\frac{1}{g(x)}}$ or $\dfrac{g(x)}{\frac{1}{f(x)}}$ which are of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x \to a$ and can be evaluated by using L' Hospital's Rule.

The form $0 . (-\infty)$ is also evaluated in the same manner.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\underset{x \to 0}{\text{Lt}} \sin x \log x^2$

Sol. $\underset{x \to 0}{\text{Lt}} (\sin x \log x^2)$ $(0 . \infty \text{ form})$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log x^2}{\frac{1}{\sin x}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log x^2}{\text{cosec}\, x} = \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{x^2} . 2x}{-\text{cosec}\, x \cot x} = -2 \underset{x \to 0}{\text{Lt}} \left(\frac{\sin^2 x}{x^2} . \frac{x}{\cos x}\right)$$

$$= -2 \underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^2 . \underset{x \to 0}{\text{Lt}} \frac{1}{\cos x} . \underset{x \to 0}{\text{Lt}} x = -2 (1) \times \frac{1}{1} \times 0 = 0.$$

Example 2. Evaluate $\underset{x \to 1}{\text{Lt}} (1 - x) \tan \frac{\pi x}{2}$.

Sol. $\underset{x \to 1}{\text{Lt}} (1 - x) \tan \frac{\pi x}{2}$ $(0 . \infty \text{ form})$

$$= \underset{x \to 1}{\text{Lt}} \frac{1 - x}{\cot \frac{\pi x}{2}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 1}{\text{Lt}} \frac{-1}{\left(-\text{cosec}^2 \frac{\pi x}{2}\right) \times \frac{\pi}{2}} = \frac{2}{\pi} . \sin^2 \frac{\pi}{2} = \frac{2}{\pi} \times (1) = \frac{2}{\pi}.$$

Example 3. Evaluate the following limits :

(i) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (1 - \sin x) \tan x$ (ii) $\underset{x \to 1}{\text{Lt}} \sec \frac{\pi x}{2} \log \frac{1}{x}$

(iii) $\underset{x \to 0}{\text{Lt}} x \tan \left(\frac{\pi}{2} - x\right)$ (iv) $\underset{x \to c}{\text{Lt}} (c - x) \tan \left(\frac{\pi x}{2c}\right)$

(v) $\underset{x \to \infty}{\text{Lt}} x \tan \frac{1}{x}$ (vi) $\underset{x \to \infty}{\text{Lt}} 2^x \sin \left(\frac{a}{2^x}\right)$.

Sol. (i) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (1 - \sin x) \tan x$ $(0 . \infty \text{ form})$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{1 - \sin x}{\cot x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{-\cos x}{-\operatorname{cosec}^2 x} = \frac{\cos \frac{\pi}{2}}{\operatorname{cosec}^2 \frac{\pi}{2}} = \frac{0}{1} = 0.$$

(*ii*) $\underset{x \to 1}{\text{Lt}} \sec \frac{\pi x}{2} \log \frac{1}{x}$ $\qquad (0 \,.\, \infty \text{ form})$

$$= \underset{x \to 1}{\text{Lt}} \frac{\log \frac{1}{x}}{\cos \frac{\pi x}{2}} = \underset{x \to 1}{\text{Lt}} \frac{-\log x}{\cos \frac{\pi x}{2}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 1}{\text{Lt}} \frac{-\frac{1}{x}}{-\frac{\pi}{2} \sin \frac{\pi x}{2}} = \frac{-\frac{1}{1}}{-\frac{\pi}{2} \sin \frac{\pi}{2}} = \frac{1}{\frac{\pi}{2} \times 1} = \frac{2}{\pi}.$$

(*iii*) $\underset{x \to 0}{\text{Lt}} x \tan \left(\frac{\pi}{2} - x\right)$ $\qquad (0 \,.\, \infty \text{ form})$

$$= \underset{x \to 0}{\text{Lt}} x \cot x = \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} = 1.$$

(*iv*) $\underset{x \to c}{\text{Lt}} (c - x) \tan \left(\frac{\pi x}{2c}\right)$ $\qquad (0.\ \infty \text{ form})$

$$= \underset{x \to c}{\text{Lt}} \frac{c - x}{\cot \left(\frac{\pi x}{2c}\right)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to c}{\text{Lt}} \frac{-1}{-\frac{\pi}{2c} \operatorname{cosec}^2 \left(\frac{\pi x}{2c}\right)} = \frac{-1}{-\frac{\pi}{2c} \operatorname{cosec}^2 \frac{\pi}{2}} = \frac{2c}{\pi}.$$

(*v*) $\underset{x \to \infty}{\text{Lt}} x \tan \frac{1}{x}$ $\qquad$ Put $x = \frac{1}{h}$, $h > 0$ so that $h \to 0+$ as $x \to \infty$

$$= \underset{h \to 0+}{\text{Lt}} \frac{1}{h} \tan h = \underset{h \to 0+}{\text{Lt}} \left(\frac{\tan h}{h}\right) = 1.$$

(*vi*) $\underset{x \to \infty}{\text{Lt}} 2^x \sin \left(\frac{a}{2^x}\right)$ $\qquad (0 \,.\, \infty \text{ form})$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\sin \left(\frac{a}{2^x}\right)}{\frac{1}{2^x}} \qquad \text{Put } x = \frac{1}{h},\ h > 0 \text{ so that } h \to 0+ \text{ as } x \to \infty$$

$$= \underset{h \to 0+}{\text{Lt}} \left[\frac{\sin \left(\frac{a}{2^{\frac{1}{h}}}\right)}{\frac{1}{2^{\frac{1}{h}}}}\right] = \underset{h \to 0+}{\text{Lt}} \frac{\cos \left(\frac{a}{2^{\frac{1}{h}}}\right) . a . 2^{\frac{-1}{h}} . \log 2 . \left(\frac{1}{h^2}\right)}{2^{\frac{-1}{h}} \log 2 . \left(\frac{1}{h^2}\right)}$$

$$= \underset{h \to 0+}{\text{Lt}} a \cos \left(a\, 2^{-\frac{1}{h}}\right) = a \cos 0 = a \times 1 = a.$$

Example 4. Evaluate $\underset{x \to 0+}{\text{Lt}} x^m . \log x$, where $m > 0$.

Sol. $\underset{x \to 0+}{\text{Lt}} x^m . \log x$ $\qquad (0 \,.\, \infty \text{ form})$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log x}{x^{-m}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{x}}{-m x^{-m-1}} = -\frac{1}{m} \underset{x \to 0+}{\text{Lt}} x^m = 0.$$

Example 5. Evaluate the following limits :

(*i*) $\underset{x \to 0+}{\text{Lt}} x \log x$ $\qquad$ (*ii*) $\underset{x \to 0+}{\text{Lt}} x^m (\log x)^n$, $m, n \in \mathbb{N}$

(P.U. 2006, 2008)

(*iii*) $\underset{x \to 0+}{\text{Lt}} x \log \tan x$ $\qquad$ (*iv*) $\underset{x \to 0+}{\text{Lt}} \sin x . \log x$

Sol. (*i*) $\underset{x \to 0+}{\text{Lt}} x \log x$ $\qquad (0 \,.\, \infty \text{ form})$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log x}{x^{-1}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{x}}{-x^{-2}} = -\underset{x \to 0+}{\text{Lt}} x = 0.$$

(*ii*) $\underset{x \to 0+}{\text{Lt}} x^m (\log x)^n$ $\qquad (0 \,.\, \infty \text{ form})$

$$= \underset{x \to 0+}{\text{Lt}} \frac{(\log x)^n}{x^{-m}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{n(\log x)^{n-1} \cdot \frac{1}{x}}{-m x^{-m-1}} = (-1)^1 \frac{n}{m} \underset{x \to 0+}{\text{Lt}} \frac{(\log x)^{n-1}}{x^{-m}}$$

$$= (-1)^1 \cdot \frac{n}{m} \underset{x \to 0+}{\text{Lt}} \frac{(n-1)(\log x)^{n-2} \cdot \frac{1}{x}}{-m x^{-m-1}}$$

$$= (-1)^2 \frac{n(n-1)}{m^2} \underset{x \to 0+}{\text{Lt}} \frac{(\log x)^{n-2}}{x^{-m}}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$= \frac{(-1)^n n(n-1)\ldots 2.1}{m^n} \underset{x \to 0+}{\text{Lt}} \frac{(\log x)^{n-n}}{x^{-m}}$$

$$= \frac{(-1)^n \underline{|n}}{m^n} \underset{x \to 0+}{\text{Lt}} \frac{1}{x^{-m}} = \frac{(-1)^n \underline{|n}}{m^n} \underset{x \to 0+}{\text{Lt}} x^m$$

$$= \frac{(-1)^n \underline{|n}}{m^n} \times 0 \qquad \left[\because \underset{x \to 0+}{\text{Lt}} x^m = 0,\ m \in N\right]$$

$$= 0$$

(*iii*) $\underset{x \to 0+}{\text{Lt}} x \cdot \log(\tan x)$ $\qquad (0 \cdot \infty \text{ form})$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log(\tan x)}{\frac{1}{x}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{-\frac{1}{x^2}} = -\underset{x \to 0+}{\text{Lt}} \left[\left(\frac{x}{\tan x}\right) \cdot (x \sec^2 x)\right]$$

$$= -\underset{x \to 0+}{\text{Lt}} \frac{x}{\tan x} \times \underset{x \to 0+}{\text{Lt}} x \sec^2 x$$

$$= -1 \times 0 = 0.$$

(*iv*) $\underset{x \to 0+}{\text{Lt}} \sin x \cdot \log x$ $\qquad (0 \cdot \infty \text{ form})$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log x}{\operatorname{cosec} x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} = -\underset{x \to 0+}{\text{Lt}} \left[\left(\frac{\sin x}{x}\right) \cdot (\tan x)\right]$$

$$= -\underset{x \to 0+}{\text{Lt}} \frac{\sin x}{x} \cdot \underset{x \to 0+}{\text{Lt}} \tan x$$

$$= -1 \times 0 = 0.$$

Art-7. Indeterminate Forms $0^0, 1^\infty, \infty^0$

Here we are to evaluate $\underset{x \to a}{\text{Lt}} [f(x)]^{g(x)}$, when

(*i*) $\underset{x \to a}{\text{Lt}} f(x) = 0 = \underset{x \to a}{\text{Lt}} g(x)$ $\qquad$ (*ii*) $\underset{x \to a}{\text{Lt}} f(x) = 1,\ \underset{x \to a}{\text{Lt}} g(x) = \infty$

(*iii*) $\underset{x \to a}{\text{Lt}} f(x) = \infty,\ \underset{x \to a}{\text{Lt}} g(x) = 0$

We proceed like this :

Let $y = [f(x)]^{g(x)}$

Taking logs on both sides,

$$\log y = \log [f(x)]^{g(x)} \Rightarrow \log y = g(x) \cdot \log [f(x)]$$

Now $g(x) \cdot \log f(x)$ becomes of the form $0 \cdot \infty$ when $x \to a$ and can be evaluated.

Let $\underset{x \to a}{\text{Lt}} g(x) \cdot \log f(x) = l$

$$\therefore \underset{x \to a}{\text{Lt}} \log y = l \qquad \Rightarrow \log \underset{x \to a}{\text{Lt}} y = l$$

$$\Rightarrow \underset{x \to a}{\text{Lt}} y = e^l \qquad \Rightarrow \underset{x \to a}{\text{Lt}} [f(x)]^{g(x)}] = e^l.$$

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\underset{x \to 0+}{\text{Lt}} x^x$.

Sol. Let $y = x^x$

$$\therefore \log y = \log x^x \Rightarrow \log y = x \cdot \log x$$

$$\therefore \underset{x \to 0+}{\text{Lt}} \log y = \underset{x \to 0+}{\text{Lt}} x \cdot \log x \qquad (0 \cdot \infty \text{ form})$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log x}{\frac{1}{x}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \underset{x \to 0+}{\text{Lt}} (-x) = 0$$

$$\therefore \log \underset{x \to 0+}{\text{Lt}} y = 0 \qquad \Rightarrow \underset{x \to 0+}{\text{Lt}} y = e^0$$

$$\Rightarrow \underset{x \to 0+}{\text{Lt}} x^x = 1.$$

Example 2. Evaluate $\underset{x \to 0}{\text{Lt}} (\sin x)^{\tan x}$. $\qquad$ (H.P.U. 2006; P.U. 2007)

Sol. Let $y = (\sin x)^{\tan x}$

$\therefore \log y = \log (\sin x)^{\tan x} \Rightarrow \log y = \tan x \,.\, \log (\sin x)$

$\therefore \underset{x \to 0}{\text{Lt}} \log y = \underset{x \to 0}{\text{Lt}} [\tan x \,.\, \log (\sin x)] = \underset{x \to 0}{\text{Lt}} \dfrac{\log (\sin x)}{\cot x} \quad \left(\dfrac{\infty}{\infty} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\frac{1}{\sin x}.\cos x}{-\text{cosec}^2 x} = -\underset{x \to 0}{\text{Lt}} (\sin x \cos x) = -(0 \times 1)$

$\because \underset{x \to 0}{\text{Lt}} \log y = 0 \quad \Rightarrow \log \underset{x \to 0}{\text{Lt}} y = 0$

$\Rightarrow \underset{x \to 0}{\text{Lt}} y = e^0 \quad \Rightarrow \underset{x \to 0}{\text{Lt}} (\sin x)^{\tan x} = 1.$

Example 3. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{x^2}\right)^{\tan x}$.

Sol. Let $y = \left(\dfrac{1}{x^2}\right)^{\tan x}$

$\therefore \log y = \log \left(\dfrac{1}{x^2}\right)^{\tan x} \Rightarrow \log y = \tan x [\log 1 - \log x^2]$

$\Rightarrow \log y = \tan x [0 - 2 \log x] \Rightarrow \log y = -2 \tan x \log x$

$\therefore \underset{x \to 0}{\text{Lt}} \log y = -2 \underset{x \to 0}{\text{Lt}} \tan x \,.\, \log x \quad (0 \,.\, \infty \text{ form})$

$= -2 \underset{x \to 0}{\text{Lt}} \dfrac{\log x}{\cot x} \quad \left(\dfrac{\infty}{\infty} \text{ form}\right)$

$= -2 \underset{x \to 0}{\text{Lt}} \dfrac{\frac{1}{x}}{-\text{cosec}^2 x} = 2 \underset{x \to 0}{\text{Lt}} \dfrac{\sin^2 x}{x} = 2 \underset{x \to 0}{\text{Lt}} \left(\dfrac{\sin x}{x}\right)^2 .x$

$= 2 (1)^2 \,.\, 0$

$\therefore \underset{x \to 0}{\text{Lt}} \log y = 0 \quad \Rightarrow \log \underset{x \to 0}{\text{Lt}} y = 0$

$\Rightarrow \underset{x \to 0}{\text{Lt}} y = e^0 \quad \Rightarrow \underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{x^2}\right)^{\tan x} = 1.$

Example 4. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$. (G.N.D.U. 2006; P.U. 2006)

Sol. Let $y = \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$

$\therefore \log y = \log \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}} \Rightarrow \log y = \dfrac{1}{x} \log \left(\dfrac{\sin x}{x}\right)$

$\therefore \underset{x \to 0}{\text{Lt}} \log y = \underset{x \to 0}{\text{Lt}} \dfrac{1}{x} \log \left(\dfrac{\sin x}{x}\right) \quad (0 \,.\, \infty \text{ form})$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\log \left(\frac{\sin x}{x}\right)}{x} \quad \left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\frac{x}{\sin x} \times \frac{x \cos x - \sin x}{x^2}}{1} = \underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \sin x}{x \sin x}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \sin x}{x^2 \left(\frac{\sin x}{x}\right)} = \underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \sin x}{x^2}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{-x \sin x + \cos x - \cos x}{2x} = -\underset{x \to 0}{\text{Lt}} \dfrac{x \sin x}{2x}$

$= -\dfrac{1}{2} \underset{x \to 0}{\text{Lt}} \sin x = -\dfrac{1}{2} \times 0 = 0$

$\therefore \underset{x \to 0}{\text{Lt}} \log y = 0 \quad \Rightarrow \log \underset{x \to 0}{\text{Lt}} y = 0$

$\therefore \underset{x \to 0}{\text{Lt}} y = e^0 \quad \Rightarrow \underset{x \to 0}{\text{Lt}} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}} = 1.$

Example 5. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$. (P.U. 2002; H.P.U. 2008)

Sol. Let $y = \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$\therefore \log y = \log \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}} \Rightarrow \log y = \dfrac{1}{x^2} \log \left(\dfrac{\sin x}{x}\right)$

$\therefore \underset{x \to 0}{\text{Lt}} \log y = \underset{x \to 0}{\text{Lt}} \dfrac{1}{x^2} .\log \left(\dfrac{\sin x}{x}\right) \quad (0 \,.\, \infty \text{ form})$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{x}{\sin x} \times \frac{x \cos x - \sin x}{x^2}}{2x} = \underset{x \to 0}{\text{Lt}} \frac{x \cos x - \sin x}{2x^2 \sin x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x \cos x - \sin x}{2x^3\left(\frac{\sin x}{x}\right)} = \frac{1}{2} \underset{x \to 0}{\text{Lt}} \frac{x \cos x - \sin x}{x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{1}{2} \underset{x \to 0}{\text{Lt}} \frac{-x \sin x + \cos x - \cos x}{3x^2} = -\frac{1}{6} \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = -\frac{1}{6} (1)$$

$\therefore \underset{x \to 0}{\text{Lt}} \log y = -\frac{1}{6} \quad \Rightarrow \log \underset{x \to 0}{\text{Lt}} y = -\frac{1}{6}$

$\Rightarrow \underset{x \to 0}{\text{Lt}} y = e^{-\frac{1}{6}} \quad \Rightarrow \underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}.$

Example 6. Evaluate $\underset{x \to 0}{\text{Lt}} \left[\tan\left(\frac{\pi}{4} + x\right)\right]^{\frac{1}{x}}$

Sol. Let $y = \left[\tan\left(\frac{\pi}{4} + x\right)\right]^{\frac{1}{x}}$

$\therefore \log y = \log\left[\tan\left(\frac{\pi}{4} + x\right)\right]^{\frac{1}{x}} = \frac{1}{x} \log\left[\tan\left(\frac{\pi}{4} + x\right)\right]$

$$\therefore \underset{x \to 0}{\text{Lt}} \log y = \underset{x \to 0}{\text{Lt}} \frac{\log\left[\tan\left(\frac{\pi}{4} + x\right)\right]}{x} = \underset{x \to 0}{\text{Lt}} \frac{\frac{\sec^2\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} + x\right)}}{1}$$

$$= \underset{x \to 0}{\text{Lt}} \left[\frac{1}{\cos^2\left(\frac{\pi}{4} + x\right)} \cdot \frac{\cos\left(\frac{\pi}{4} + x\right)}{\sin\left(\frac{\pi}{4} + x\right)}\right] = \underset{x \to 0}{\text{Lt}} \left[\frac{2}{2 \sin\left(\frac{\pi}{4 + x}\right) \cos\left(\frac{\pi}{4} + x\right)}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2}{\sin\left(\frac{\pi}{2} + 2x\right)} = \underset{x \to 0}{\text{Lt}} \frac{2}{\cos 2x} = \frac{2}{1}$$

$\therefore \underset{x \to 0}{\text{Lt}} \log y = 2 \quad \Rightarrow \log \underset{x \to 0}{\text{Lt}} y = 2$

$\Rightarrow \underset{x \to 0}{\text{Lt}} y = e^2 \quad \Rightarrow \underset{x \to 0}{\text{Lt}} \left[\tan\left(\frac{\pi}{4} + 4\right)\right]^{\frac{1}{x}} = e^2$

Example 7. Evaluate the following limits :

(*i*) $\underset{x \to 0+}{\text{Lt}} x^{\sin x}$ (*ii*) $\underset{x \to 0+}{\text{Lt}} x^{\frac{1}{\log x}}$

(*iii*) $\underset{x \to a+}{\text{Lt}} (x - a)^{x - a}$ (*iv*) $\underset{x \to 1-}{\text{Lt}} (1 - x^2)^{\frac{1}{\log(1-x)}}$

(*v*) $\underset{x \to 0+}{\text{Lt}} (\tan x)^{\sin 2x}$ (*vi*) $\underset{x \to \frac{\pi}{2}-}{\text{Lt}} (\cos x)^{\cos x}$

Sol. (*i*) Let $y = x^{\sin x}$.

$\therefore \log y = \log x^{\sin x} \Rightarrow \log y = \sin x \cdot \log x$

$\therefore \underset{x \to 0+}{\text{Lt}} \log y = \underset{x \to 0+}{\text{Lt}} \sin x \cdot \log x \qquad (0 \cdot \infty \text{ form})$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log x}{\operatorname{cosec} x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} = -\underset{x \to 0+}{\text{Lt}} \left(\frac{\sin x}{x} \cdot \tan x\right)$$

$$= -\underset{x \to 0+}{\text{Lt}} \frac{\sin x}{x} \cdot \underset{x \to 0+}{\text{Lt}} \tan x = -1 \times 0 = 0$$

$\therefore \log \underset{x \to 0+}{\text{Lt}} y = 0 \Rightarrow \underset{x \to 0+}{\text{Lt}} y = e^0 \Rightarrow \underset{x \to 0+}{\text{Lt}} x^{\sin x} = 1.$

(*ii*) Let $y = x^{\frac{1}{\log x}}$

$\therefore \log y = \log x^{\frac{1}{\log x}} = \frac{1}{\log x} \cdot \log x = 1$

$\therefore \underset{x \to 0+}{\text{Lt}} \log y = 1 \Rightarrow \log \underset{x \to 0+}{\text{Lt}} y = 1 \Rightarrow \underset{x \to 0+}{\text{Lt}} y = e^1$

$\Rightarrow \underset{x \to 0+}{\text{Lt}} x^{\frac{1}{\log x}} = e.$

(iii) Let $y = (x-a)^{x-a}$

$\therefore \log y = \log (x-a)^{x-a} \Rightarrow \log y = (x-a)\log(x-a)$

$\therefore \lim_{x \to a+} \log y = \lim_{x \to a+} (x-a).\log(x-a)$ (0 . ∞ form)

$$= \lim_{x \to a+} \frac{\log(x-a)}{\frac{1}{x-a}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to a+} \frac{\frac{1}{x-a}}{-\frac{1}{(x-a)^2}} = -\lim_{x \to a+} (x-a) = 0$$

$\therefore \lim_{x \to a+} \log y = 0$

$\Rightarrow \log \lim_{x \to a+} y = 0 \Rightarrow \lim_{x \to a+} y = e^0 \Rightarrow \lim_{x \to a+} (x-a)^{x-a} = 1.$

(iv) Let $y = (1-x^2)^{\frac{1}{\log(1-x)}}$

$\therefore \log y = \log (1-x^2)^{\frac{1}{\log(1-x)}} = \frac{1}{\log(1-x)} \log(1-x^2)$

$\therefore \lim_{x \to 1-} \log y = \lim_{x \to 1-} \frac{\log(1-x^2)}{\log(1-x)}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x \to 1-} \frac{\frac{-2x}{1-x^2}}{-\frac{1}{1-x}} = \lim_{x \to 1-} \frac{2x}{1+x} = \frac{2}{1+1} = 1$$

$\therefore \lim_{x \to 1-} \log y = 1 \Rightarrow \log \lim_{x \to 1-} y = 1 \Rightarrow \lim_{x \to 1-} y = e^1$

$\Rightarrow \lim_{x \to 1-} (1-x^2)^{\frac{1}{\log(1-x)}} = e.$

(v) Let $y = (\tan x)^{\sin 2x}$

$\therefore \log y = \log (\tan x)^{\sin 2x} \Rightarrow \log y = \sin 2x . \log \tan x$

$\therefore \lim_{x \to 0+} \log y = \lim_{x \to 0+} \sin 2x . \log \tan x$ (0 . ∞ form)

$$= \lim_{x \to 0+} \frac{\log \tan x}{\operatorname{cosec} 2x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0+} \frac{\frac{1}{\tan x}.\sec^2 x}{-2 \operatorname{cosec} 2x \cot 2x} = -\frac{1}{2} \lim_{x \to 0+} \frac{\cos x}{\sin x}.\frac{1}{\cos^2 x}.\frac{\sin^2 2x}{\cos 2x}$$

$$= -\frac{1}{2} \lim_{x \to 0+} \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x} \times \sin 2x . \frac{\sin 2x}{\cos 2x}$$

$$= -\frac{1}{2} \lim_{x \to 0+} \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x} . 2 \sin x . \cos x \tan 2x$$

$$= -\lim_{x \to 0+} (\tan 2x) = 0$$

$\therefore \lim_{x \to 0+} \log y = 0 \Rightarrow \log \lim_{x \to 0+} y = 0 \Rightarrow \lim_{x \to 0+} y = e^0$

$\Rightarrow \lim_{x \to 0+} (\tan x)^{\sin 2x} = 1.$

(vi) Let $y = (\cos x)^{\cos x}$

$\therefore \log y = \log (\cos x)^{\cos x} = \cos x . \log \cos x$

$\therefore \lim_{x \to \frac{\pi}{2}-} \log y = \lim_{x \to \frac{\pi}{2}-} \cos x . \log \cos x$ (0 . ∞ form)

$$= \lim_{x \to \frac{\pi}{2}-} \frac{\log \cos x}{\sec x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}-} \frac{\frac{1}{\cos x}.(-\sin x)}{\sec x \tan x} = -\lim_{x \to \frac{\pi}{2}-} \cos x = 0$$

$\therefore \lim_{x \to \frac{\pi}{2}-} \log y = 0 \Rightarrow \log \lim_{x \to \frac{\pi}{2}-} y = 0 \Rightarrow \lim_{x \to \frac{\pi}{2}-} y = e^0$

$\Rightarrow \lim_{x \to \frac{\pi}{2}-} (\cos x)^{\cos x} = 1.$

Example 8. Evaluate the following limits :

(i) $\lim_{x \to \infty} (1+x)^{\frac{1}{x}}$ (ii) $\lim_{x \to 0+} (\cot x)^x$ (H.P.U. 2006)

(iii) $\lim_{x \to 0+} (\cot x)^{\sin x}$ (iv) $\lim_{x \to \frac{\pi}{2}-} (\tan x)^{\sin 2x}$

(H.P.U. 2007)

(v) $\lim_{x \to 0+} \left(\frac{1}{x}\right)^{\tan x}$ (vi) $\lim_{x \to 0+} (\cot x)^{\frac{1}{\log x}}$

(vii) $\lim_{x \to \frac{\pi}{2}-} (\sec x)^{\cot x}$ (viii) $\lim_{x \to 0+} (\cot x)^{\sin 2x}$.

Sol. (i) Let $y = (1+x)^{\frac{1}{x}}$

$\therefore \log y = \log (1+x)^{\frac{1}{x}} \Rightarrow \log y = \frac{1}{x} \log (1+x)$

$\therefore \lim_{x \to \infty} \log y = \lim_{x \to \infty} \frac{\log (1+x)}{x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x \to \infty} \frac{\frac{1}{1+x}}{1} = \lim_{x \to \infty} \frac{1}{1+x} = 0$$

$\therefore \lim_{x \to \infty} \log y = 0 \Rightarrow \log \lim_{x \to \infty} y = 0 \Rightarrow \lim_{x \to \infty} y = e^0$.

$\Rightarrow \lim_{x \to \infty} (1+x)^{\frac{1}{x}} = 1.$

(ii) Let $y = (\cot x)^x$

$\therefore \log y = \log (\cot x)^x = x \,.\, \log \cot x$

$\therefore \lim_{x \to 0+} \log y = \lim_{x \to 0+} x \,.\, \log \cot x$ $(0 \,.\, \infty \text{ form})$

$$= \lim_{x \to 0+} \frac{\log \cot x}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0+} \frac{\frac{1}{\cot x}.(-\operatorname{cosec}^2 x)}{-\frac{1}{x^2}} = \lim_{x \to 0+} \frac{\sin x}{\cos x} \cdot \frac{x^2}{\sin^2 x}$$

$$= \lim_{x \to 0+} \tan x \,.\left(\frac{x}{\sin x}\right)^2 = 0 \times 1 = 0$$

$\therefore \lim_{x \to 0+} \log y = 0 \Rightarrow \log \lim_{x \to 0+} y = 0 \Rightarrow \lim_{x \to 0+} y = e^0$

$\Rightarrow \lim_{x \to 0+} (\cot x)^x = 1.$

(iii) Let $y = (\cot x)^{\sin x}$

$\therefore \log y = \log (\cot x)^{\sin x} = \sin x \,.\, \log (\cot x)$

$\therefore \lim_{x \to 0+} \log y = \lim_{x \to 0+} \sin x \,.\, \log (\cot x)$ $(0 \,.\, \infty \text{ form})$

$$= \lim_{x \to 0+} \frac{\log (\cot x)}{\operatorname{cosec} x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0+} \frac{\frac{1}{\cot x}.(-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x} = \lim_{x \to 0+} \frac{\operatorname{cosec} x}{\cot^2 x}$$

$$= \lim_{x \to 0+} \frac{\frac{1}{\sin x}}{\frac{\cos^2 x}{\sin^2 x}} = \lim_{x \to 0+} \frac{\sin x}{\cos^2 x} = \frac{0}{1} = 0$$

$\therefore \lim_{x \to 0+} \log y = 0 \Rightarrow \log \lim_{x \to 0+} y = 0 \Rightarrow \lim_{x \to 0+} y = e^0$

$\Rightarrow \lim_{x \to 0+} (\cot x)^{\sin x} = 1.$

(iv) Let $y = (\tan x)^{\sin 2x}$

$\therefore \log y = \log (\tan x)^{\sin 2x} = \sin 2x \,.\, \log (\tan x)$

$\therefore \lim_{x \to \frac{\pi}{2}-} \log y = \lim_{x \to \frac{\pi}{2}-} \sin 2x \,.\, \log (\tan x)$ $(0 \,.\, \infty \text{ form})$

$$= \lim_{x \to \frac{\pi}{2}-} \frac{\log (\tan x)}{\operatorname{cosec} 2x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}-} \frac{\frac{1}{\tan x}.\sec^2 x}{-2\operatorname{cosec} 2x \cot 2x}$$

$$= -\frac{1}{2} \lim_{x \to \frac{\pi}{2}-} \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} . \sin 2x \frac{\sin 2x}{\cos 2x}$$

$$= -\frac{1}{2} \lim_{x \to \frac{\pi}{2}-} \frac{1}{\sin x} \cdot \frac{1}{\cos x} \,.\, 2 \sin x \cos x \,.\, \tan 2x$$

$$= -\lim_{x \to \frac{\pi}{2}-} \tan 2x = 0$$

$\therefore \lim_{x \to \frac{\pi}{2}-} \log y = 0 \Rightarrow \log \lim_{x \to \frac{\pi}{2}-} y = 0 \Rightarrow \lim_{x \to \frac{\pi}{2}-} y = e^0$

$\Rightarrow \lim_{x \to \frac{\pi}{2}-} (\tan x)^{\sin 2x} = 1.$

(v) Let $y = \left(\frac{1}{x}\right)^{\tan x}$

$\therefore \quad \log y = \log\left(\frac{1}{x}\right)^{\tan x} = \tan x \log \frac{1}{x} = -\tan x \log x$

$\therefore \quad \underset{x \to 0+}{\text{Lt}} \log y = -\underset{x \to 0+}{\text{Lt}} \tan x \, . \log x \qquad (0 \, . \, \infty \text{ form})$

$= -\underset{x \to 0+}{\text{Lt}} \frac{\log x}{\cot x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$

$= -\underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{x}}{-\text{cosec}^2 x} = \underset{x \to 0+}{\text{Lt}} \frac{\sin^2 x}{x}$

$= \underset{x \to 0+}{\text{Lt}} \frac{2 \sin x \cos x}{1} = 2 \sin 0 \cos 0 = 2 \times 0 \times 1 = 0$

$\therefore \quad \underset{x \to 0+}{\text{Lt}} \log y = 0 \Rightarrow \log \underset{x \to 0+}{\text{Lt}} y = 0 \Rightarrow \underset{x \to 0+}{\text{Lt}} y = e^0$

$\Rightarrow \quad \underset{x \to 0+}{\text{Lt}} \left(\frac{1}{x}\right)^{\tan x} = 1.$

(vi) Put $y = (\cot x)^{\frac{1}{\log x}}$

$\therefore \quad \log y = \log (\cot x)^{\frac{1}{\log x}} \Rightarrow \log y = \frac{1}{\log x} . \log (\cot x)$

$\therefore \quad \underset{x \to 0+}{\text{Lt}} \log y = \underset{x \to 0+}{\text{Lt}} \frac{1}{\log x} . \log (\cot x) \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$

$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{\cot x} . (-\text{cosec}^2 x)}{\frac{1}{x}} = -\underset{x \to 0+}{\text{Lt}} \left(\frac{x}{\sin x} . \frac{1}{\cos x}\right)$

$= -\underset{x \to 0+}{\text{Lt}} \frac{x}{\sin x} . \underset{x \to 0+}{\text{Lt}} \frac{1}{\cos x} = -1 \times \frac{1}{1} = -1$

$\therefore \quad \underset{x \to 0+}{\text{Lt}} \log y = -1 \Rightarrow \log \underset{x \to 0+}{\text{Lt}} y = -1 \Rightarrow \underset{x \to 0+}{\text{Lt}} y = e^{-1}$

$\Rightarrow \quad \underset{x \to 0+}{\text{Lt}} (\cot x)^{\frac{1}{\log x}} = \frac{1}{e}.$

(vii) Put $y = (\sec x)^{\cot x}$

$\therefore \quad \log y = \log (\sec x)^{\cot x} \Rightarrow \log y = \cot x \, . \log (\sec x)$

$\therefore \quad \underset{x \to \frac{\pi}{2}-}{\text{Lt}} \log y = \underset{x \to \frac{\pi}{2}-}{\text{Lt}} \cot x \, . \log (\sec x)$

$= \underset{x \to \frac{\pi}{2}-}{\text{Lt}} \frac{\log (\sec x)}{\tan x} = \underset{x \to \frac{\pi}{2}-}{\text{Lt}} \frac{\frac{1}{\sec x} . \sec x \tan x}{\sec^2 x}$

$= \underset{x \to \frac{\pi}{2}-}{\text{Lt}} \left(\frac{\sin x}{\cos x} . \cos^2 x\right) = \underset{x \to \frac{\pi}{2}-}{\text{Lt}} (\sin x \cos x)$

$= \sin \frac{\pi}{2} \cos \frac{\pi}{2} = 1 \times 0 = 0$

$\therefore \quad \underset{x \to \frac{\pi}{2}-}{\text{Lt}} \log y = 0 \Rightarrow \log \underset{x \to \frac{\pi}{2}-}{\text{Lt}} y = 0 \Rightarrow \underset{x \to \frac{\pi}{2}-}{\text{Lt}} y = e^0$

$\Rightarrow \quad \underset{x \to \frac{\pi}{2}-}{\text{Lt}} (\sec x)^{\cot x} = 1.$

(viii) Put $y = (\cot x)^{\sin 2x}$

$\therefore \quad \log y = \log (\cot x)^{\sin 2x} \Rightarrow \log y = \sin 2x \, . \log (\cot x)$

$\therefore \quad \underset{x \to 0+}{\text{Lt}} \log y = \underset{x \to 0+}{\text{Lt}} \sin 2x \, . \log (\cot x)$

$= \underset{x \to 0+}{\text{Lt}} \frac{\log (\cot x)}{\text{cosec } 2x} = \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{\cot x}(-\text{cosec}^2 x).}{-2 \, \text{cosec } 2x \cot 2x}$

$= \frac{1}{2} \underset{x \to 0+}{\text{Lt}} \left[\frac{\sin x}{\cos x} \times \frac{1}{\sin^2 x} \times \sin 2x . \frac{\sin 2x}{\cos 2x}\right]$

$= \frac{1}{2} \underset{x \to 0+}{\text{Lt}} \left[\frac{1}{\cos x} \times \frac{1}{\sin x} \times 2 \sin x \cos x \times \tan 2x\right]$

$= \underset{x \to 0+}{\text{Lt}} \tan 2x = \tan 0 = 0$

$\therefore \quad \underset{x \to 0+}{\text{Lt}} \log y = 0$

$\Rightarrow \quad \log \underset{x \to 0+}{\text{Lt}} y = 0 \Rightarrow \underset{x \to 0+}{\text{Lt}} y = e^0 \Rightarrow \underset{x \to 0+}{\text{Lt}} (\cot x)^{\sin 2x} = 1.$